ROMANIAN MATHEMATICAL MAGAZINE

**If $a, b, c \in \mathbb{R}$ and $(a+1)(b+1)(c+1) = 8$, then prove that :**

$$a^2 + b^2 + c^2 \geq 3$$

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We assert that $\forall\, x, y, z \geq 0$, we have : $\sqrt[3]{xyz} \leq \dfrac{x+y+z}{3} \to (1)$

When $x = y = z = 0$, then trivially $(1)$ is true. When exactly two among $x, y, z$ $= 0$ and WLOG we may assume $y = z = 0$ $(x > 0)$, then :

$$\frac{x+y+z}{3} = \frac{x}{3} > 0 = \sqrt[3]{xyz} \Rightarrow (1) \text{ is true}$$

When exactly one among $x, y, z = 0$ and WLOG we may assume $x = 0$ $(y, z > 0)$, then : $\dfrac{x+y+z}{3} = \dfrac{y+z}{3} > 0 = \sqrt[3]{xyz} \Rightarrow (1)$ is true. When $x, y, z > 0$, $(1)$ is true by AM $-$ GM and so, $\forall\, x, y, z \geq 0$, we have : $\sqrt[3]{xyz} \leq \dfrac{x+y+z}{3}$

Now, $(a+1)(b+1)(b+1) = 8 \Rightarrow (a+1)^2(b+1)^2(c+1)^2 = 64 \Rightarrow 4$ $= \sqrt[3]{(a+1)^2(b+1)^2(c+1)^2} \overset{\text{via (1)}}{\leq} \dfrac{(a+1)^2 + (b+1)^2 + (c+1)^2}{3}$

$$\Rightarrow \sum_{\text{cyc}} a^2 + 2\sum_{\text{cyc}} a \geq 9 \to (2)$$

**Case 1** $\displaystyle\sum_{\text{cyc}} a \leq 3$ and so, via $(2)$, $\displaystyle\sum_{\text{cyc}} a^2 \geq 9 - 2\sum_{\text{cyc}} a \geq 9 - 6 \Rightarrow a^2 + b^2 + c^2 \geq 3$

**Case 2** $\displaystyle\sum_{\text{cyc}} a \geq 3$ and then : $\displaystyle\sum_{\text{cyc}} a^2 \geq \frac{1}{3}\left(\sum_{\text{cyc}} a\right)^2 \left(\Leftrightarrow \sum_{\text{cyc}} (a-b)^2 \geq 0 \to \text{true}\right)$ $\geq \dfrac{9}{3} \Rightarrow a^2 + b^2 + c^2 \geq 3$ $\therefore$ combining both cases, $a^2 + b^2 + c^2 \geq 3$ $\forall\, a, b, c \in \mathbb{R} \,\big|\, (a+1)(b+1)(c+1) = 8$ (QED)