ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $abc = 1$, then prove that :

$$\frac{a}{b^{2024}} + \frac{b}{c^{2024}} + \frac{c}{a^{2024}} \ge a + b + c$$

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$$\frac{a}{b^{2024}} + \frac{b}{c^{2024}} + \frac{c}{a^{2024}} = \frac{\left(\frac{1}{b}\right)^{2024}}{\frac{1}{a}} + \frac{\left(\frac{1}{c}\right)^{2024}}{\frac{1}{b}} + \frac{\left(\frac{1}{a}\right)^{2024}}{\frac{1}{c}} \overset{\text{Holder}}{\ge} \frac{\left(\sum_{cyc}\frac{1}{a}\right)^{2024}}{3^{2022}\left(\sum_{cyc}\frac{1}{a}\right)}$$

$$= \frac{\left(\sum_{cyc}\frac{1}{a}\right)^{2}\left(\sum_{cyc}\frac{1}{a}\right)^{2021}}{3^{2022}} \overset{A-G}{\ge} \frac{\frac{\left(\sum_{cyc}ab\right)^2}{a^2b^2c^2}\cdot\left(3\sqrt[3]{\frac{1}{abc}}\right)^{2021}}{3^{2022}} \ge \frac{\frac{3abc\sum_{cyc}a}{a^2b^2c^2}}{3} \ (\because abc = 1)$$

$$= \frac{\sum_{cyc}a}{abc} \overset{abc=1}{=} \sum_{cyc} a \therefore \frac{a}{b^{2024}} + \frac{b}{c^{2024}} + \frac{c}{a^{2024}} \ge a + b + c$$

$$\forall\, a, b, c > 0 \,\middle|\, abc = 1, ''='' \text{ iff } a = b = c = 1 \text{ (QED)}$$