ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a^2 + b^2 + c^2 \leq 192$, then prove that :

$$\sqrt{a^3+64}+\sqrt{b^3+64}+\sqrt{b^3+64}\leq 72$$

Proposed by Nguyen Hung Cuong

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sqrt{a^3+64}+\sqrt{b^3+64}+\sqrt{b^3+64}=\sum_{cyc}\sqrt{(a+4)(a^2-4a+16)}$$

$$\overset{CBS}{\leq}\sqrt{\sum_{cyc}(a+4)}.\sqrt{\sum_{cyc}(a^2-4a+16)}=\sqrt{t+12}.\sqrt{\sum_{cyc}a^2-4t+48}\left(t=\sum_{cyc}a\right)$$

$$\overset{a^2+b^2+c^2\leq 192}{\leq}\sqrt{(t+12)(240-4t)}\left(\begin{array}{c}\text{note}: 240-4t\overset{CBS}{\geq}240-4.\sqrt{3\sum_{cyc}a^2}\\ \overset{a^2+b^2+c^2\leq 192}{\geq}240-96>0\end{array}\right)$$

$$\overset{?}{\leq}72\Leftrightarrow t^2-48t+576\overset{?}{\geq}0\Leftrightarrow(t-24)^2\overset{?}{\geq}0\to \text{true}$$

$$\therefore\sqrt{a^3+64}+\sqrt{b^3+64}+\sqrt{b^3+64}\leq 72$$

$$\forall\, a,b,c>0 \text{ and } a^2+b^2+c^2\leq 192\ (\text{QED})$$

Solution 2 by Tapas Das-India

Let $a^2=p, b^2=q, c^2=r$, *and* $p+q+r\leq 192$ *and* $\sum\sqrt{a^3+64}=\sum\sqrt{p^{\frac{3}{2}}+64}$,

now we will show as a lemma that $\dfrac{32+p}{4}\geq\sqrt{p^{\frac{3}{2}}+64}$ *or*,

$$(32+p)^2\geq 16\left(p^{\frac{3}{2}}+64\right) \text{ or } u^2-16u+64\overset{p=u^2}{\geq}0 \text{ or } (u-8)^2\geq 0(True).$$

back to the main problem:

$$LHS\overset{lemma}{\leq}\sum\frac{32+p}{4}=\frac{96+p+q+r}{4}\overset{\sum p\leq 192}{\leq}=\frac{96+192}{4}=72$$