ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$, then prove that :

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a^2+b^2} \geq \frac{24(a^2+b^2)}{(a+b)^4}$$

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$$(a^2+b^2+2ab)^2 \overset{A-G}{\geq} 8ab(a^2+b^2) \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a^2+b^2}-\frac{24(a^2+b^2)}{(a+b)^4}$$

$$\geq \frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a^2+b^2}-\frac{24(a^2+b^2)}{8ab(a^2+b^2)} = \frac{1}{a^2}+\frac{1}{b^2}-\frac{2}{ab}-\left(\frac{1}{ab}-\frac{2}{a^2+b^2}\right)$$

$$= \frac{a^2+b^2-2ab}{a^2b^2}-\frac{a^2+b^2-2ab}{ab(a^2+b^2)} = \frac{(a-b)^2(a^2+b^2-ab)}{a^2b^2(a^2+b^2)} \overset{A-G}{\geq}$$

$$\frac{(a-b)^2(2ab-ab)}{a^2b^2(a^2+b^2)} = \frac{(a-b)^2}{ab(a^2+b^2)} \geq 0 \therefore \frac{1}{a^2}+\frac{1}{b^2}+\frac{2}{a^2+b^2} \geq \frac{24(a^2+b^2)}{(a+b)^4}$$

$$\forall\, a, b > 0, '' ='' \text{ iff } a = b \text{ (QED)}$$