ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, a^2+b^2+c^2=abc$ then:

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\le\frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Tapas Das-India

$$\frac{a}{a^2+bc}+\frac{b}{b^2+ca}+\frac{c}{c^2+ab}\overset{AM-HM}{\le}\frac{1}{4}\sum\left(\frac{a}{a^2}+\frac{a}{bc}\right)=\frac{1}{4}\sum\frac{1}{a}+\frac{1}{4}\frac{a^2+b^2+c^2}{abc}$$

$$=\frac{1}{4}\frac{ab+bc+ca}{abc}+\frac{1}{4}\le\frac{1}{4}\frac{\sum a^2}{abc}+\frac{1}{4}=\frac{1}{2}\left(since\sum a^2=abc\right)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{LHS}=\sum_{\text{cyc}}\frac{a^2+\text{bc}-\text{bc}}{a(a^2+\text{bc})}=\sum_{\text{cyc}}\frac{1}{a}-\sum_{\text{cyc}}\frac{\text{b}^2\text{c}^2}{a^3\text{bc}+a\text{b}^2\text{c}^2}\overset{\text{Bergstrom}}{\le}$$

$$\frac{\sum_{\text{cyc}}ab}{abc}-\frac{\left(\sum_{\text{cyc}}ab\right)^2}{abc\sum_{\text{cyc}}a^2+abc\sum_{\text{cyc}}ab}\le\frac{\sum_{\text{cyc}}ab}{abc}-\frac{\left(\sum_{\text{cyc}}ab\right)^2}{abc\sum_{\text{cyc}}a^2+abc\sum_{\text{cyc}}a^2}$$

$$\overset{a^2+\text{b}^2+\text{c}^2=abc}{=}\frac{\sum_{\text{cyc}}ab}{abc}-\frac{\left(\sum_{\text{cyc}}ab\right)^2}{2a^2\text{b}^2\text{c}^2}=\frac{2abc\sum_{\text{cyc}}ab-\left(\sum_{\text{cyc}}ab\right)^2}{2a^2\text{b}^2\text{c}^2}\Rightarrow\text{LHS}-\text{RHS}\le$$

$$\frac{2abc\sum_{\text{cyc}}ab-\left(\sum_{\text{cyc}}ab\right)^2}{2a^2\text{b}^2\text{c}^2}-\frac{1}{2}=\frac{2abc\sum_{\text{cyc}}ab-\left(\sum_{\text{cyc}}ab\right)^2-a^2\text{b}^2\text{c}^2}{2a^2\text{b}^2\text{c}^2}$$

$$=-\frac{1}{2a^2\text{b}^2\text{c}^2}\cdot\left(abc-\sum_{\text{cyc}}ab\right)^2=\frac{-1}{2a^2\text{b}^2\text{c}^2}\cdot\left(\sum_{\text{cyc}}a^2-\sum_{\text{cyc}}ab\right)^2\le 0$$

$$\therefore\frac{a}{a^2+\text{bc}}+\frac{\text{b}}{\text{b}^2+\text{c}a}+\frac{\text{c}}{\text{c}^2+ab}\le\frac{1}{2}$$

$$\forall\, a,\text{b},\text{c}>0\,\middle|\,a^2+\text{b}^2+\text{c}^2=abc,''=''\text{ iff } a=\text{b}=\text{c}=3\ (\text{QED})$$