ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $ab + bc + ca = 3$, then prove that :

$$\frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(c + a)} + \frac{1}{1 + c^2(a + b)} \le \frac{1}{abc}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$3 = ab + bc + ca \overset{A-G}{\ge} 3\sqrt[3]{a^2b^2c^2} \Rightarrow abc \le 1 \to \text{(i)}$$

$$\text{Now, LHS} = \sum_{\text{cyc}} \frac{1}{1 + a(3 - bc)} \; (\because a(b + c) = 3 - bc \text{ and analogs})$$

$$= \sum_{\text{cyc}} \frac{1}{1 - abc + 3a} \overset{\text{via (i)}}{\le} \sum_{\text{cyc}} \frac{1}{3a} = \frac{1}{3abc} . \sum_{\text{cyc}} ab = \frac{3}{3abc}$$

$$\therefore \frac{1}{1 + a^2(b + c)} + \frac{1}{1 + b^2(c + a)} + \frac{1}{1 + c^2(a + b)} \le \frac{1}{abc}$$

$$\forall\, a, b, c > 0 \,\big|\, ab + bc + ca = 3, '' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$