ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0$ and $a+b+c=3$, then prove that :

$$\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\geq 3$$

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$$\left(\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\right)^2\geq$$

$$\sum_{cyc}a+\sum_{cyc}(b-c)^2+2\sum_{cyc}\sqrt{ab}=\sum_{cyc}a+2\sum_{cyc}a^2-2\sum_{cyc}ab+2\sum_{cyc}\sqrt{ab}\overset{?}{\geq}9$$

$$\Leftrightarrow\sum_{cyc}x^2+2\sum_{cyc}x^4-2\sum_{cyc}x^2y^2+2\sum_{cyc}xy\overset{?}{\geq}9\ (\sqrt{a}=x,\sqrt{b}=y,\sqrt{c}=z)$$

$$\Leftrightarrow\frac{1}{3}\left(\sum_{cyc}x^2\right)^2+2\sum_{cyc}x^4-2\sum_{cyc}x^2y^2+\frac{2}{3}\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2\right)\overset{?}{\geq}\left(\sum_{cyc}x^2\right)^2$$

$$\left(\because\sum_{cyc}x^2=3\right)\Leftrightarrow\sum_{cyc}x^4+2\sum_{cyc}x^2y^2+6\sum_{cyc}x^4-6\sum_{cyc}x^2y^2$$

$$+2\left(\sum_{cyc}x^3y+\sum_{cyc}xy^3+xyz\sum_{cyc}x\right)\overset{?}{\geq}3\sum_{cyc}x^4+6\sum_{cyc}x^2y^2$$

$$\Leftrightarrow 2\sum_{cyc}x^4-5\sum_{cyc}x^2y^2+\sum_{cyc}x^3y+\sum_{cyc}xy^3+xyz\sum_{cyc}x\underset{(*)}{\overset{?}{\geq}}0$$

Now, via Schur, $\sum_{cyc}x^4+xyz\sum_{cyc}x\geq\sum_{cyc}x^3y+\sum_{cyc}xy^3\Rightarrow$ LHS of $(*)\geq$

$$\sum_{cyc}x^4+2\sum_{cyc}x^3y+2\sum_{cyc}xy^3-5\sum_{cyc}x^2y^2\geq\sum_{cyc}x^4+4\sum_{cyc}x^2y^2-5\sum_{cyc}x^2y^2$$

$$(\because x^3y+xy^3-2x^2y^2=xy(x-y)^2\geq 0\Rightarrow x^3y+xy^3\geq 2x^2y^2\text{ and analogs})$$

$$=\sum_{cyc}x^4-\sum_{cyc}x^2y^2=\frac{1}{2}\sum_{cyc}(x^2-y^2)^2\geq 0\Rightarrow(*)\text{ is true}$$

$$\therefore\left(\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\right)^2\geq 9$$

$$\Rightarrow\sqrt{a+(b-c)^2}+\sqrt{b+(c-a)^2}+\sqrt{c+(a-b)^2}\geq 3$$

$$\forall a,b,c>0\,\big|\,a+b+c=3,''='' \text{ iff } a=b=c=1\ (QED)$$