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If $a, \mathrm{b}, \mathrm{c} > 0$ and $a\mathrm{bc} \geq 1$, then prove that :

$$a+\mathrm{b}+\mathrm{c} \geq \frac{1+a}{1+\mathrm{b}}+\frac{1+\mathrm{b}}{1+\mathrm{c}}+\frac{1+\mathrm{c}}{1+a}$$

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$$a+\mathrm{b}+\mathrm{c} \geq \frac{1+a}{1+\mathrm{b}}+\frac{1+\mathrm{b}}{1+\mathrm{c}}+\frac{1+\mathrm{c}}{1+a}$$

$$\Leftrightarrow a\left(1-\frac{1}{1+\mathrm{b}}\right)-\frac{1}{1+\mathrm{b}}+\mathrm{b}\left(1-\frac{1}{1+\mathrm{c}}\right)-\frac{1}{1+\mathrm{c}}+\mathrm{c}\left(1-\frac{1}{1+a}\right)-\frac{1}{1+a} \geq 0$$

$$\Leftrightarrow \frac{a\mathrm{b}-1}{1+\mathrm{b}}+\frac{\mathrm{bc}-1}{1+\mathrm{c}}+\frac{\mathrm{c}a-1}{1+a} \geq 0 \Leftrightarrow$$

$$(a\mathrm{b}-1)(1+\mathrm{c})(1+a)+(\mathrm{bc}-1)(1+a)(1+\mathrm{b})+(\mathrm{c}a-1)(1+\mathrm{b})(1+\mathrm{c}) \geq 0$$

$$\Leftrightarrow a\mathrm{bc}\sum_{\mathrm{cyc}} a+3a\mathrm{bc}-3+\sum_{\mathrm{cyc}} a^2\mathrm{b}-2\sum_{\mathrm{cyc}} a \overset{(*)}{\geq} 0$$

$$\because a\mathrm{bc} \geq 1 \therefore a\mathrm{bc}\sum_{\mathrm{cyc}} a-\sum_{\mathrm{cyc}} a+3a\mathrm{bc}-3 \geq 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : } \sum_{\mathrm{cyc}} a^2\mathrm{b} \overset{(**)}{\geq} \sum_{\mathrm{cyc}} a$$

$$\boxed{\text{Case 1}} \sum_{\mathrm{cyc}} a \geq \sum_{\mathrm{cyc}} a\mathrm{b} \text{ and then : } \sum_{\mathrm{cyc}} a^2\mathrm{b}=\sum_{\mathrm{cyc}} \frac{a^2}{\frac{1}{\mathrm{b}}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\mathrm{cyc}} a\right)^2}{\sum_{\mathrm{cyc}} \frac{1}{a}}$$

$$=a\mathrm{bc}.\frac{\left(\sum_{\mathrm{cyc}} a\right)^2}{\sum_{\mathrm{cyc}} a\mathrm{b}} \overset{\substack{\sum_{\mathrm{cyc}} a\mathrm{b} \leq \sum_{\mathrm{cyc}} a \\ \text{and} \\ \because a\mathrm{bc} \geq 1}}{\geq} \frac{\left(\sum_{\mathrm{cyc}} a\right)^2}{\sum_{\mathrm{cyc}} a}=\sum_{\mathrm{cyc}} a \Rightarrow \sum_{\mathrm{cyc}} a^2\mathrm{b} \geq \sum_{\mathrm{cyc}} a$$

$$\boxed{\text{Case 2}} \sum_{\mathrm{cyc}} a\mathrm{b} \geq \sum_{\mathrm{cyc}} a \text{ and then : } \sum_{\mathrm{cyc}} a^2\mathrm{b}=\sum_{\mathrm{cyc}} \frac{a^2\mathrm{b}^2}{\mathrm{b}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sum_{\mathrm{cyc}} a\mathrm{b}\right)^2}{\sum_{\mathrm{cyc}} a}$$

$$\overset{\sum_{\mathrm{cyc}} a\mathrm{b} \geq \sum_{\mathrm{cyc}} a}{\geq} \frac{\left(\sum_{\mathrm{cyc}} a\right)^2}{\sum_{\mathrm{cyc}} a}=\sum_{\mathrm{cyc}} a \Rightarrow \sum_{\mathrm{cyc}} a^2\mathrm{b} \geq \sum_{\mathrm{cyc}} a \therefore \text{combining both cases,}$$

$$(**) \Rightarrow (*) \text{ is true } \forall\, a,\mathrm{b},\mathrm{c}>0 \,\middle|\, a\mathrm{bc} \geq 1 \Rightarrow a+\mathrm{b}+\mathrm{c} \geq \frac{1+a}{1+\mathrm{b}}+\frac{1+\mathrm{b}}{1+\mathrm{c}}+\frac{1+\mathrm{c}}{1+a}$$

$$\forall\, a,\mathrm{b},\mathrm{c}>0 \,\middle|\, a\mathrm{bc} \geq 1, ''='' \text{ iff } a=\mathrm{b}=\mathrm{c}=1 \text{ (QED)}$$