ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2\ge 3$, then prove that:

$$\frac{a^2}{b+2c}+\frac{b^2}{c+2a}+\frac{c^2}{a+2b}\ge 1$$

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$$\frac{a^2}{b+2c}+\frac{b^2}{c+2a}+\frac{c^2}{a+2b}=\frac{a^4}{a^2b+2ca^2}+\frac{b^4}{b^2c+2ab^2}+\frac{c^4}{c^2a+2bc^2}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b+\sum_{cyc}ab^2+\sum_{cyc}\left(a\left(\sum_{cyc}a^2-c^2-a^2\right)\right)}$$

$$=\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}a^2b+\sum_{cyc}ab^2+\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)-\sum_{cyc}a^2b-\sum_{cyc}a^3}\overset{\sum_{cyc}ab^2\le\sum_{cyc}a^3}{\ge}$$

$$\frac{\left(\sum_{cyc}a^2\right)^2}{\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)}=\frac{\sum_{cyc}a^2}{\sum_{cyc}a}=\frac{\sqrt{\frac{\sum_{cyc}a^2}{3}}\cdot\sqrt{3\sum_{cyc}a^2}}{\sum_{cyc}a}\overset{\substack{a^2+b^2+c^2\ge 3\\ \text{and}\\ 3\sum_{cyc}a^2\ge\left(\sum_{cyc}a\right)^2}}{\ge}=1$$

$$\therefore\frac{a^2}{b+2c}+\frac{b^2}{c+2a}+\frac{c^2}{a+2b}\ge 1\ \forall\, a,b,c>0\,\middle|\,a^2+b^2+c^2\ge 3,$$

$$''='' \text{ iff } a=b=c=1\ (\text{QED})$$