ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} \geq 0$ and $a + \mathrm{b} + \mathrm{c} = 1$, then prove that :

$$ab + bc + ca \geq 8(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 among a, b, c equal zero and WLOG we may assume $b = c = 0$ $(a = 1)$ and then : $LHS = RHS = 0$

$\boxed{\text{Case 2}}$ Exactly 1 among a, b, c equals zero and WLOG we may assume $a = 0$ $(b + c = 1$ with $b, c > 0)$ and then : $LHS - RHS \overset{b+c=1}{=}$

$$bc(b+c)^4 - 8b^2c^2(b^2+c^2) = bc\left((b^2+c^2+2bc)^2 - 8bc(b^2+c^2)\right)$$

$$= bc\left((b^2+c^2)^2 + 4b^2c^2 - 4bc(b^2+c^2)\right) = bc(b^2+c^2-2bc)^2 = bc(b-c)^4 \geq 0$$

$$\Rightarrow LHS \geq RHS$$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow$ $x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z$, $y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say); so $2\sum_{cyc} a = \sum_{cyc} x = 2s$

$$\Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z \therefore abc = r^2 s \to (2)$$

and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3)$,

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4), \sum_{cyc} a^2b^2 = \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right)$$

$$\overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s \Rightarrow \sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2) \to (5)$$

$$\therefore ab + bc + ca \geq 8(a^2 + b^2 + c^2)(a^2b^2 + b^2c^2 + c^2a^2) \overset{a+b+c=1}{\Leftrightarrow}$$

$$\left(\sum_{cyc} a\right)^4 \left(\sum_{cyc} ab\right) \geq 8\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)$$

$$\overset{\text{via (1),(3),(4) and (5)}}{\Leftrightarrow} (4Rr + r^2)s^4 \geq 8r^2(s^2 - 8Rr - 2r^2)((4R + r)^2 - 2s^2)$$

$$\Leftrightarrow (4R+17r)s^4 - rs^2(128R^2+192Rr+40r^2)+16r^2(4R+r)^3 \overset{(*)}{\geq} 0 \text{ and}$$
$$\because (4R+17r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*),$$
$$\text{it suffices to prove : LHS of } (*) \geq (4R+17r)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (312R-210r)s^2 \overset{(**)}{\geq} r(2944R^2-2812Rr+409r^2)$$
$$\text{Now, } (312R-210r)s^2 \overset{\text{Gerretsen}}{\geq} (312R-210r)(16Rr-5r^2) \overset{?}{\geq}$$
$$r(2944R^2-2812Rr+409r^2) \Leftrightarrow 2048R^2-2108Rr+641r^2 \overset{?}{\geq} 0$$
$$\Leftrightarrow 994R^2+1054R(R-2r)+641r^2 \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore ab+bc+ca > 8(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2) \text{ and combining all cases,}$$
$$ab+bc+ca \geq 8(a^2+b^2+c^2)(a^2b^2+b^2c^2+c^2a^2)\ \forall\, a,b,c \geq 0 \,\big|\, a+b+c=1,$$
$$'' = ''\ \text{ iff } (a=1, b=c=0) \text{ and permutations and } \left(a=0, b=c=\frac{1}{2}\right)$$
$$\text{and permutations (QED)}$$