ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} > 0$ and $a^2 + \mathrm{b}^2 \geq 2$, then prove that :

$$\frac{a^2}{\mathrm{b}} + \frac{\mathrm{b}^2}{a} + 7(a + \mathrm{b}) \geq 16$$

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$$\text{Let } x = a + \mathrm{b} \text{ and } \mathrm{y} = a\mathrm{b} \therefore a^2 + \mathrm{b}^2 \geq 2 \Leftrightarrow x^2 - 2\mathrm{y} \geq 2$$

$$\Leftrightarrow 2\mathrm{y} \leq x^2 - 2 \to (\mathrm{i})$$

$$\text{Now, } \frac{a^2}{\mathrm{b}} + \frac{\mathrm{b}^2}{a} + 7(a + \mathrm{b}) = \frac{(a + \mathrm{b})(a^2 - a\mathrm{b} + \mathrm{b}^2)}{a\mathrm{b}} + 7(a + \mathrm{b}) = \frac{x(x^2 - 3\mathrm{y})}{\mathrm{y}} + 7x$$

$$= \frac{x^3 + 4x\mathrm{y}}{\mathrm{y}} \overset{?}{\geq} 16 \Leftrightarrow x^3 \underbrace{\overset{?}{\geq}}_{(*)} 4\mathrm{y}(4 - x)$$

$\boxed{\text{Case 1}}$ $4 - x \leq 0$ and $\because x, \mathrm{y} > 0 \therefore$ *LHS of* $(*) > 0$ and RHS of $(*) \leq 0 \Rightarrow$ LHS of $(*) >$ RHS of $(*)$

$\boxed{\text{Case 2}}$ $4 - x > 0$ and then : $4\mathrm{y}(4 - x) \overset{\text{via (i)}}{\leq} (8 - 2x)(x^2 - 2) \overset{?}{\leq} x^3 \Leftrightarrow$

$$3x^3 - 8x^2 - 4x + 16 \overset{?}{\geq} 0 \Leftrightarrow (3x + 4)(x - 2)^2 \overset{?}{\geq} 0 \to \text{true} \because x = a + \mathrm{b} > 0$$

$\Rightarrow (*)$ is true $\therefore$ combining both cases, $(*)$ is true $\forall\, x, \mathrm{y} > 0$ *constrained* by (i)

$$\therefore \frac{a^2}{\mathrm{b}} + \frac{\mathrm{b}^2}{a} + 7(a + \mathrm{b}) \geq 16 \;\forall\, a, \mathrm{b} > 0 \,\big|\, a^2 + \mathrm{b}^2 \geq 2 \text{ (QED)}$$