ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } a,b,c \geq -\frac{3}{2} \text{ and } abc + ab + bc + ca + a + b + c \geq 0,$$
$$\text{then prove that} : a + b + c \geq 0$$

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$$\text{Let } a + \frac{3}{2} = x, b + \frac{3}{2} = y, c + \frac{3}{2} = z \text{ and then} : a = \frac{2x-3}{2}, b = \frac{2y-3}{2},$$
$$c = \frac{2z-3}{2} \Rightarrow abc + ab + bc + ca + a + b + c \geq 0$$
$$\Leftrightarrow \frac{1}{8}\prod_{\text{cyc}}(2x-3) + \frac{1}{4}\sum_{\text{cyc}}\left((2x-3)(2y-3)\right) + \frac{1}{2}\sum_{\text{cyc}}(2x-3) \geq 0$$
$$\Leftrightarrow 2\sum_{\text{cyc}} x + 8xyz \geq 4\sum_{\text{cyc}} xy + 9 \rightarrow (1) \text{ and we are to prove} : \frac{1}{2}\sum_{\text{cyc}}(2x-3) \geq 0$$
$$\Leftrightarrow 2\sum_{\text{cyc}} x \geq 9 \rightarrow (*)$$
$$\boxed{\text{Case 1}}\ 2xyz \geq \sum_{\text{cyc}} xy \text{ and } \because x,y,z \geq 0 \therefore 2xyz \geq \sum_{\text{cyc}} xy \overset{A-G}{\geq} 3\sqrt[3]{x^2y^2z^2}$$
$$\Rightarrow 8x^3y^3z^3 \geq 27x^2y^2z^2 \Rightarrow xyz \geq \frac{27}{8} \Rightarrow \sqrt[3]{xyz} \geq \frac{3}{2} \therefore \sum_{\text{cyc}} x \overset{A-G}{\geq} 3\sqrt[3]{xyz} \geq \frac{9}{2}$$
$$\Rightarrow (*) \text{ is true}$$
$$\boxed{\text{Case 2}}\ \sum_{\text{cyc}} xy \geq 2xyz \text{ and then, via } (1), 2\sum_{\text{cyc}} x - 9 \geq 4\left(\sum_{\text{cyc}} xy - 2xyz\right) \geq 0$$
$$\Rightarrow (*) \text{ is true} \therefore \text{combining both cases, } (*) \text{ is true } \forall\, x,y,z \geq 0 \text{ constrained by } (1)$$
$$\therefore a + b + c \geq 0\ \forall\, a,b,c \geq -\frac{3}{2}\ \Big|\ abc + ab + bc + ca + a + b + c \geq 0 \text{ (QED)}$$