ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z \in [-1,1]$ and $x+y+z+xyz=0$, then prove that :

$$\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq 3$$

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$$x+y+z+xyz=0 \Rightarrow x(1+yz)=-y-z \Rightarrow x=-\frac{y+z}{1+yz}$$

$$\Rightarrow x+1=1-\frac{y+z}{1+yz}=\frac{1-y-z(1-y)}{1+yz}\Rightarrow x+1=\frac{(1-y)(1-z)}{1+yz}\to(1)$$

$\boxed{\text{Case 1}}$ $xyz \geq 0$ and then : $\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\overset{\text{CBS}}{\leq}\sqrt{3}.\sqrt{\sum_{\text{cyc}}x+3}$

$$\overset{x+y+z+xyz=0}{=}\sqrt{3}.\sqrt{3-xyz}\overset{xyz\geq 0}{\leq}\sqrt{3}.\sqrt{3}\therefore\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq 3$$

$\boxed{\text{Case 2}}$ $xyz \leq 0$ and then : either $\boxed{\text{Case 2i}}$ $x, y, z \leq 0$ so that : $\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq\sqrt{1}+\sqrt{1}+\sqrt{1}\therefore\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq 3$

or $\boxed{\text{Case 2ii}}$ one variable ≤ 0 and the other two ≥ 0 and WLOG we may assume $x\leq 0; y,z\geq 0$ and $\because m^2+4m+4\geq 4m+4\ \forall\ m\geq 0$

$$\therefore m+1\leq\frac{(m+2)^2}{4}\Rightarrow\sqrt{m+1}\leq 1+\frac{m}{2}\text{ and so, }\sqrt{y+1}+\sqrt{z+1}\leq 1+\frac{y}{2}+1+\frac{z}{2}$$

$$\Rightarrow\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\leq\sqrt{\frac{(1-y)(1-z)}{1+yz}}+2+\frac{y+z}{2}$$

$$\left(\text{note : }\frac{(1-y)(1-z)}{1+yz}\geq 0\because 0\leq y,z\leq 1\right)$$

$$\Rightarrow\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}-3\leq\sqrt{\frac{(1-y)(1-z)}{1+yz}}+\frac{y+z}{2}-1$$

$$=\sqrt{\frac{(1-y)(1-z)}{1+yz}}-\frac{(1-y)+(1-z)}{2}\overset{A-G}{\leq}$$

$$\sqrt{\frac{(1-y)(1-z)}{1+yz}}-\sqrt{(1-y)(1-z)}\ \left(\begin{matrix}\because(1-y),(1-z)\geq 0\\ \text{as }0\leq y,z\leq 1\end{matrix}\right)$$

$$=\sqrt{(1-y)(1-z)}.\left(\frac{1}{\sqrt{1+yz}}-1\right)=\sqrt{(1-y)(1-z)}.\frac{\frac{1}{1+yz}-1}{\frac{1}{\sqrt{1+yz}}+1}$$

$$=-\sqrt{(1-y)(1-z)}.\frac{yz}{(1+yz)\left(\frac{1}{\sqrt{1+yz}}+1\right)}\leq 0\because 0\leq y,z\leq 1$$

$$\therefore \sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\le 3 \therefore \text{combining } all \text{ } cases,$$

$$\sqrt{x+1}+\sqrt{y+1}+\sqrt{z+1}\le 3 \ \forall \ x,y,z\in[-1,1] \Big| \ x+y+z+xyz=0,$$

$$''=''\ \text{ iff } x=y=z=0 \ (\text{QED})$$