ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ and $a^2+b^2+a+b=ab$, then prove that :

$$a^3+b^3 \geq -16$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = a+b \text{ and then : } a^2+b^2+a+b = ab \Rightarrow -x = a^2+b^2-ab$$
$$\geq \frac{(a+b)^2}{4} = \frac{x^2}{4} \Rightarrow x^2+4x \leq 0 \Rightarrow x(x+4) \leq 0 \Rightarrow -4 \leq x \leq 0 \to (1)$$
$$\text{Now, } a^3+b^3 = (a+b)^3 - 3ab(a+b) \overset{a^2+b^2+a+b=ab}{=} x^3 - 3x(a^2+b^2+a+b)$$
$$\geq x^3 - 3x\left(\frac{(a+b)^2}{2}+a+b\right)\left(\because -3x \geq 0 \text{ and } a^2+b^2 \geq \frac{(a+b)^2}{2}\right) \overset{?}{\geq} -16$$
$$\Leftrightarrow x^3 - 3x\left(\frac{x^2}{2}+x\right) \overset{?}{\geq} -16 \Leftrightarrow x^3+6x^2-32 \overset{?}{\leq} 0 \Leftrightarrow (x+4)^2(x-2) \overset{?}{\leq} 0 \to \text{true}$$
$$\because x \overset{\text{via }(1)}{\leq} 0 < 2 \Rightarrow x-2 < 0, '' ='' \text{ iff } x = a+b = -4$$
$$\text{and for } a = b \Rightarrow \text{iff } a = b = -2 \therefore a^3+b^3 \geq -16$$
$$\forall\, a, b \in \mathbb{R} \,\middle|\, a^2+b^2+a+b = ab, '' ='' \text{ iff } a = b = -2 \text{ (QED)}$$