ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ and $a^5 + b^5 = a^2 + b^2$, then prove that :

$$a^3 + b^3 \le 2$$

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$$a^4 - a^3b + a^2b^2 - ab^3 + b^4 = a^3(a-b) - b^3(a-b) + a^2b^2 =$$

$$(a-b)^2(a^2+b^2+ab) + a^2b^2 = (a-b)^2\left(\frac{3}{4}(a+b)^2 + \frac{1}{4}(a-b)^2\right) + a^2b^2 \ge 0$$

$$\Rightarrow a^4 - a^3b + a^2b^2 - ab^3 + b^4 \ge 0 \Rightarrow (a^2+b^2)^2 - a^2b^2 - ab(a^2+b^2) \ge 0$$

$$\Rightarrow x^2 - y^2 - xy \overset{(1)}{\ge} 0 \; (x = a^2+b^2, y = ab)$$

$$\text{Now}, a^5 + b^5 = a^2 + b^2 \Rightarrow (a+b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4) = a^2 + b^2$$

$$\Rightarrow (a+b)(x^2 - y^2 - xy) \overset{(*)}{=} x \text{ and } (1) \text{ suggests 2 cases :}$$

$\boxed{\text{Case (i)}}$ $x^2 - y^2 - xy = 0$ and then : via $(*)$, $x = 0$ and plugging $x = 0$ in $x^2 - y^2 - xy = 0$, we get : $y = 0 \therefore a^2 + b^2 = ab = 0 \Rightarrow (a+b)(a^2+b^2-ab) = 0 \Rightarrow a^3 + b^3 = 0 \therefore a^3 + b^3 < 2$

$\boxed{\text{Case (ii)}}$ $x^2 - y^2 - xy > 0$ and then : via $(*)$, $a + b = \dfrac{x}{x^2 - y^2 - xy} \Rightarrow$

$$a^3 + b^3 - 2 = (a+b)(a^2+b^2-ab) - 2 = \frac{x(x-y)}{x^2-y^2-xy} - 2 = -\frac{x^2 - xy - 2y^2}{x^2-y^2-xy}$$

$$= -\frac{(x-2y)(x+y)}{x^2-y^2-xy} = -\frac{(a^2+b^2-2ab)(a^2+b^2+ab)}{x^2-y^2-xy} =$$

$$-\frac{(a-b)^2\left(\frac{3}{4}(a+b)^2 + \frac{1}{4}(a-b)^2\right)}{x^2-y^2-xy} \le 0 \therefore a^3 + b^3 \le 2 \text{ and combining both cases,}$$

$$a^3 + b^3 \le 2 \; \forall \, a, b \in \mathbb{R} \,\middle|\, a^5 + b^5 = a^2 + b^2, \text{"="} \text{ iff } a = b = 1 \text{ (QED)}$$