ROMANIAN MATHEMATICAL MAGAZINE

If $a + b \neq 0$, then :

$$a^2 + b^2 + \left(\frac{1 - ab}{a + b}\right)^2 \geq 1$$

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$$\forall\, x, y, z \in \mathbb{R}, \sum_{cyc}(x - y)^2 \geq 0 \Rightarrow 2\sum_{cyc} x^2 + \sum_{cyc} x^2 \geq \sum_{cyc} x^2 + 2\sum_{cyc} xy$$

$$\Rightarrow \sum_{cyc} x^2 \geq \frac{1}{3}\left(\sum_{cyc} x\right)^2 \rightarrow (1) \text{ and also}, \forall\, x, y \in \mathbb{R},$$

$$(x + y)^2 - 4xy \geq 0 \Rightarrow (x + y)^2 \geq 4xy \rightarrow (2)$$

$$\therefore a^2 + b^2 + \left(\frac{1 - ab}{a + b}\right)^2 \overset{\text{via (1)}}{\geq} \frac{1}{3}\left(a + b + \frac{1 - ab}{a + b}\right)^2 = \frac{1}{3} \cdot \frac{\left((a + b)^2 - ab + 1\right)^2}{(a + b)^2}$$

$$= \frac{1}{3} \cdot \frac{\left(a^2 + b^2 + ab + 1\right)^2}{(a + b)^2} \geq \frac{1}{3} \cdot \frac{\left(\frac{3}{4}(a + b)^2 + 1\right)^2}{(a + b)^2}$$

$$\left(\begin{array}{c}\because 4\left(a^2 + b^2 + ab\right) - 3(a + b)^2 = (a - b)^2 \geq 0\ \forall\, a, b \in \mathbb{R} \\ \Rightarrow a^2 + b^2 + ab \geq \frac{3}{4}(a + b)^2\ \forall\, a, b \in \mathbb{R}\end{array}\right) \overset{\text{via (2)}}{\geq} \frac{1}{3} \cdot \frac{4 \cdot \frac{3}{4}(a + b)^2 \cdot 1}{(a + b)^2}$$

$$= 1 \therefore a^2 + b^2 + \left(\frac{1 - ab}{a + b}\right)^2 \geq 1\ \forall\, a, b \in \mathbb{R} \,|\, a + b \neq 0,$$

$$'' ='' \text{ iff } \left(a = b = \frac{1}{\sqrt{3}}\right) \text{ or } \left(a = b = -\frac{1}{\sqrt{3}}\right) \text{ (QED)}$$