ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c = 6$, then prove that :

$$\frac{a}{\sqrt{b^3+1}} + \frac{b}{\sqrt{c^3+1}} + \frac{c}{\sqrt{a^3+1}} \geq 2$$

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We shall prove that : $\frac{1}{\sqrt{x^3+1}} \overset{(*)}{\geq} \frac{7-2x}{9}$ $\forall\, x \in (0,6)$ and we note that if $x \in \left[\frac{7}{2}, 6\right)$, then : RHS of $(*) \leq 0 <$ LHS of $(*) \Rightarrow (*)$ is true and so, we now focus on : $x \in \left(0, \frac{7}{2}\right)$ and then : $\frac{7-2x}{9} > 0 \therefore (*) \Leftrightarrow \frac{1}{x^3+1} \geq \frac{(7-2x)^2}{81}$

$$\Leftrightarrow 4x^5 - 28x^4 + 49x^3 + 4x^2 - 28x - 32 \leq 0 \Leftrightarrow (x-2)^2(4x^3 - 12x^2 - 15x - 8)$$
$$\leq 0 \Leftrightarrow (x-2)^2\left((x-4)(2x+1)^2 - 4\right) \leq 0 \to \text{true} \because x < \frac{7}{2} < 4$$

$\Rightarrow (x-4)(2x+1)^2 - 4 < 0 \therefore (*)$ is true is true and combining both cases, $(*)$ is true $\forall\, x \in (0,6)$ and via $(*)$, $\frac{a}{\sqrt{b^3+1}} + \frac{b}{\sqrt{c^3+1}} + \frac{c}{\sqrt{a^3+1}} \geq \sum_{cyc} \frac{a(7-2b)}{9}$

$$= \frac{7}{9}\sum_{cyc} a - \frac{2}{9}\sum_{cyc} ab \geq \frac{7}{9}\sum_{cyc} a - \frac{2}{27}\left(\sum_{cyc} a\right)^2 \overset{a+b+c=6}{=} \frac{7}{9}.6 - \frac{2}{27}.36 = 2$$

$$\therefore \frac{a}{\sqrt{b^3+1}} + \frac{b}{\sqrt{c^3+1}} + \frac{c}{\sqrt{a^3+1}} \geq 2$$

$\forall\, a, b, c > 0 \,\middle|\, a + b + c = 6$, " = " iff $a = b = c = 2$ (QED)