ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ and $a + b + c = 3$, then prove that :

$$a^2 + b^2 + c^2 + \frac{ab + bc + ca}{a^2b + b^2c + c^2a} \geq 4$$

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Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say);

so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \to (1) \Rightarrow a = s - x, b = s - y, c = s - z$

$\therefore abc = r^2 s \to (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3), \sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=}$$

$$s^2 - 2(4Rr + r^2) \Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4), \text{ and } \sum_{cyc} a^2b^2$$

$$= \left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2((4R + r)^2 - 2s^2) \to (5)$$

$$\text{Now, } \frac{\left(9\sum_{cyc} a^2 - 2\left(\sum_{cyc} a\right)^2\right)^2}{9\left(\sum_{cyc} a\right)^2} \overset{?}{\geq} \frac{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)}{\left(\sum_{cyc} ab\right)^2} \overset{\text{via (1),(3),(4) and (5)}}{\Leftrightarrow}$$

$$\frac{(9(s^2 - 8Rr - 2r^2) - 2s^2)^2}{9s^2} \overset{?}{\geq} \frac{r^2(s^2 - 8Rr - 2r^2)((4R + r)^2 - 2s^2)}{(4Rr + r^2)^2}$$

$$\Leftrightarrow 9s^6 + (320R^2 + 88Rr + 2r^2)s^4 - r(7488R^3 + 5616R^2r + 1404Rr^2 + 117r^3)s^2$$

$$+162r^2(4R + r)^4 \underbrace{\overset{?}{\geq}}_{(*)} 0 \text{ and } \because 9(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to}$$

prove $(*)$, it suffices to prove : LHS of $(*) \geq 9(s^2 - 16Rr + 5r^2)^3$

$$\Leftrightarrow (320R^2 + 520Rr - 133r^2)s^4 - r(7488R^3 + 12528R^2r - 2916Rr^2 + 792r^3)s^2$$

$$+r^2(41472R^4 + 78336R^3r - 19008R^2r^2 + 13392Rr^3 - 963r^4) \overset{(**)}{\geq} 0 \text{ and}$$

$$\because (320R^2 + 520Rr - 133r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order}$$

to prove $(**)$, it suffices to prove : LHS of $(**) \geq$

$$(320R^2+520Rr-133r^2)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (1376R^3+456R^2r-3270Rr^2+269r^3)s^2$$
$$\overset{(***)}{\geq} r(20224R^4+1792R^3r-45120R^2r^2+10444Rr^3-1181r^4)\ \mathit{and} \because$$
$$1376R^3+456R^2r-3270Rr^2+269r^3$$
$$=(R-2r)(1376R^2+3208Rr+3146r^2)+6561r^3 \overset{\text{Euler}}{\geq} 6561r^3>0$$
$$\therefore \text{LHS of } (***) \overset{\text{Gerretsen}}{\geq} (1376R^3+456R^2r-3270Rr^2+269r^3)(16Rr-5r^2)$$
$$\overset{?}{\geq} r(20224R^4+1792R^3r-45120R^2r^2+10444Rr^3-1181r^4)$$
$$\Leftrightarrow 896t^4-688t^3-4740t^2+5105t-82 \overset{?}{\geq} 0\ \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\Big((t-2)(896t^2+2896t+3260)+6561\Big) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \frac{9\sum_{cyc} a^2-2\left(\sum_{cyc} a\right)^2}{3\sum_{cyc} a} \geq \frac{\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)}}{\sum_{cyc} ab}$$
$$\overset{\text{Reverse CBS}}{\geq} \frac{\sum_{cyc} a^2b}{\sum_{cyc} ab} \overset{a+b+c=3}{\Rightarrow} \sum_{cyc} a^2-2 \geq \frac{\sum_{cyc} a^2b}{\sum_{cyc} ab} \Rightarrow \sum_{cyc} a^2+\frac{\sum_{cyc} ab}{\sum_{cyc} a^2b}$$
$$\geq 2+\frac{\sum_{cyc} a^2b}{\sum_{cyc} ab}+\frac{\sum_{cyc} ab}{\sum_{cyc} a^2b} \overset{A-G}{\geq} 2+2=4 \therefore a^2+b^2+c^2+\frac{ab+bc+ca}{a^2b+b^2c+c^2a} \geq 4$$
$$\forall\, a,b,c>0 \,\big|\, a+b+c=3,\ ''=''\ \text{ iff } a=b=c=1\ (\text{QED})$$