ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0$ and $a + b + c = 2$, then prove that :

$$\sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab} \leq 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$\boxed{\text{Case 1}}$ Exactly 2 variables equal zero and WLOG we may assume $b = c = 0$ $(a = 2)$ and then : $\text{LHS} = \sqrt{2^2} = 2 < 3$

$\boxed{\text{Case 2}}$ Exactly 1 variable equals zero and WLOG we may assume $a = 0$ $(b + c = 2$ with $b, c > 0)$ and then : $\text{LHS} = \sqrt{bc} + b + c \overset{A-G}{\leq} \frac{3(b+c)}{2} = 3$

$\boxed{\text{Case 3}}$ $a, b, c > 0$ and assigning $b + c = x, c + a = y, a + b = z \Rightarrow$ $x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z,$ $y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say); so $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s$ $\to (1) \Rightarrow a = s - x, b = s - y, c = s - z \therefore abc = r^2 s \to (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y) \Rightarrow \sum_{cyc} ab = 4Rr + r^2 \to (3),$

$$\sum_{cyc} a^2 = \left(\sum_{cyc} a\right)^2 - 2\sum_{cyc} ab \overset{\text{via (1) and (3)}}{=} s^2 - 2(4Rr + r^2)$$

$$\Rightarrow \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2 \to (4), \text{ and } \sum_{cyc} a^2b^2 =$$

$$\left(\sum_{cyc} ab\right)^2 - 2abc\left(\sum_{cyc} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4Rr + r^2)^2 - 2r^2 s.s$$

$$\Rightarrow \sum_{cyc} a^2b^2 = r^2\left((4R + r)^2 - 2s^2\right) \to (5)$$

$$\text{Now, } \left(\sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab}\right)^2 - 9 \overset{a+b+c=2}{=}$$

$$\sum_{cyc} a^2 + \sum_{cyc} ab + 2\sum_{cyc}\sqrt{(b^2 + ca)(c^2 + ab)} - \frac{9}{4}\left(\sum_{cyc} a\right)^2$$

$$\overset{CBS}{\leq} \sum_{cyc} a^2 + \sum_{cyc} ab + 2\sqrt{3}.\sqrt{\sum_{cyc}\left((b^2 + ca)(c^2 + ab)\right)} - \frac{9}{4}\left(\sum_{cyc} a\right)^2 \overset{?}{<} 0$$

$$\Leftrightarrow 2\sqrt{3}.\sqrt{\sum_{cyc} a^2b^2 + \sum_{cyc}\left(ab\left(\sum_{cyc} a^2 - c^2\right)\right) + abc\sum_{cyc} a} \overset{?}{<} \frac{5\sum_{cyc} a^2 + 14\sum_{cyc} ab}{4}$$

$$\Leftrightarrow 2\sqrt{3}.\sqrt{\sum_{cyc} a^2b^2 + \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)} \overset{?}{<} \frac{5\sum_{cyc} a^2 + 14\sum_{cyc} ab}{4}$$

$$\Leftrightarrow \left(5\sum_{cyc} a^2 + 14\sum_{cyc} ab\right)^2 \overset{?}{>} 192\left(\sum_{cyc} a^2b^2 + \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} ab\right)\right)$$

$$\overset{\text{via (3),(4) and (5)}}{\Leftrightarrow} \left(5(s^2 - 8Rr - 2r^2) + 14(4Rr + r^2)\right)^2$$

$$\overset{?}{>} 192\left(r^2\left((4R + r)^2 - 2s^2\right) + (s^2 - 8Rr - 2r^2)(4Rr + r^2)\right)$$

$$\Leftrightarrow 25s^4 - (608Rr - 232r^2)s^2 + r^2(3328R^2 + 1664Rr + 208r^2) \underset{(*)}{\overset{?}{>}} 0 \text{ and}$$

$$\because 25(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) > 25(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (64R - 6r)s^2 \overset{(**)}{>} r(1024R^2 - 1888Rr + 139r^2)$$

$$\text{Now, } (64R - 6r)s^2 \overset{\text{Gerretsen}}{\geq} (64R - 6r)(16Rr - 5r^2) \overset{?}{>}$$

$$r(1024R^2 - 1888Rr + 139r^2) \Leftrightarrow r^2(1472R - 109r) \overset{?}{>} 0 \to \text{true}$$

$$\because 1472R - 109r \overset{\text{Euler}}{\geq} 2944r - 1097r > 0 \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \left(\sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab}\right)^2 - 9 < 0$$

$$\Rightarrow \sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab} < 3 \ \forall\, a, b, c > 0 \,|\, a + b + c = 2$$

$$\therefore \text{combining } \textit{all} \text{ cases}, \sqrt{a^2 + bc} + \sqrt{b^2 + ca} + \sqrt{c^2 + ab} \leq 3$$

$$\forall\, a, b, c \geq 0 \,|\, a + b + c = 2, \text{''} = \text{''} \text{ iff } (a = 0, b = c = 1)$$

$$\text{or } (b = 0, c = a = 1) \text{ or } (c = 0, a = b = 1) \text{ (QED)}$$