ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ and $2(a^2 + b^2) = 3ab + 1$, then prove that :

$$a^3 + b^3 \leq 2$$

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$$\text{If } a + b \leq 0, \text{then} : a^3 + b^3 = (a + b)\left(\frac{1}{4}(a + b)^2 + \frac{3}{4}(a - b)^2\right) \leq 0 < 2$$

$$\text{and so, we now focus on} : a + b > 0 \text{ and then} : a^3 + b^3 \leq 2 \Leftrightarrow (a^3 + b^3)^2 \leq 4$$

$$\Leftrightarrow (a + b)^2(a^2 + b^2 - ab)^2 \leq 4 \Leftrightarrow (a^2 + b^2 + 2ab)(a^2 + b^2 - ab)^2 \leq 4$$

$$\overset{\because 2(a^2+b^2)=3ab+1}{\Leftrightarrow} \left(\frac{3ab + 1}{2} + 2ab\right)\left(\frac{3ab + 1}{2} - ab\right)^2 \leq 4 \Leftrightarrow (7x + 1)(x + 1)^2 \leq 32$$

$$(x = ab) \Leftrightarrow 7x^3 + 15x^2 + 9x - 31 \leq 0 \Leftrightarrow (x - 1)(7x^2 + 22x + 31) \leq 0$$

$$\Leftrightarrow x \leq 1 \left(\begin{array}{c}\because \text{discriminant of } 7x^2 + 22x + 31 = 22^2 - 4.7.31 = -384 < 0 \\ \Rightarrow 7x^2 + 22x + 31 > 0\end{array}\right)$$

$$\rightarrow \text{true} \because 3ab + 1 = 2(a^2 + b^2) \geq 4ab \Rightarrow x = ab \leq 1 \therefore a^3 + b^3 \leq 2$$

$$\forall\, a, b \in \mathbb{R} \,\middle|\, 2(a^2 + b^2) = 3ab + 1, \text{''} = \text{''} \text{ iff } a = b = 1 \text{ (QED)}$$