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If $a, b, c \in \mathbb{R}$ and $a(1+b^2)+b(1+c^2)+c(1+a^2) \geq 6$,

then prove that : $a^2+b^2+c^2 \geq 3$

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$$\text{Let us assume : } \sum_{cyc} a^2 < 3$$

$$\text{Now, } 6 \leq \sum_{cyc} a + \sum_{cyc} ab^2 \overset{CBS}{\leq} \sum_{cyc} a + \sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right)} \overset{assumption}{<}$$

$$\sum_{cyc} a + \sqrt{3\sum_{cyc} a^2b^2} \overset{\left(\sum_{cyc} a^2\right)^2 \geq 3\sum_{cyc} a^2b^2}{\leq} \sum_{cyc} a + \sum_{cyc} a^2 \overset{assumption}{<} \sum_{cyc} a + 3$$

$$\Rightarrow \sum_{cyc} a > 3 \to (*)$$

$$\text{But : } 3 > \sum_{cyc} a^2 \geq \frac{1}{3}\left(\sum_{cyc} a\right)^2 \Rightarrow \left(\sum_{cyc} a\right)^2 - 9 < 0$$

$$\Rightarrow \left(\sum_{cyc} a + 3\right)\left(\sum_{cyc} a - 3\right) < 0 \Rightarrow -3 < \sum_{cyc} a < 3 \to (**)$$

$\therefore (*), (**)$ **contradict each other $\therefore$ our assumption must be incorrect and so,**

we conclude : $a^2+b^2+c^2 \geq 3$ (QED)