ROMANIAN MATHEMATICAL MAGAZINE

If $abc = 1$, then :

$$\sqrt{\frac{2}{1+a^2}}+\sqrt{\frac{2}{1+b^2}}+\sqrt{\frac{2}{1+c^2}}\le 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Since $abc = 1$, we may assign $a=\frac{yz}{x^2}, b=\frac{zx}{y^2}, c=\frac{xy}{z^2}$ and then :

$$\sum_{cyc}\frac{1}{\sqrt{1+a^2}}=\sum_{cyc}\frac{1}{\sqrt{1+\frac{y^2z^2}{x^4}}}=\sum_{cyc}\frac{x^2}{\sqrt{x^4+y^2z^2}}$$

$$=\sum_{cyc}\frac{x^2.\sqrt{y^4+z^2x^2}.\sqrt{z^4+x^2y^2}}{\sqrt{(x^4+y^2z^2)(y^4+z^2x^2)(z^4+x^2y^2)}}$$

$$=\sum_{cyc}\frac{\left(x.\sqrt{y^4+z^2x^2}\right)\left(x.\sqrt{z^4+x^2y^2}\right)}{\sqrt{(x^4+y^2z^2)(y^4+z^2x^2)(z^4+x^2y^2)}}\overset{CBS}{\le}$$

$$\frac{1}{\sqrt{(x^4+y^2z^2)(y^4+z^2x^2)(z^4+x^2y^2)}}.\left(\begin{array}{c}\left(\sqrt{x^2(y^4+z^2x^2)+y^2(z^4+x^2y^2)+z^2(x^4+y^2z^2)}\right)\\ *\\ \left(\sqrt{x^2(z^4+x^2y^2)+y^2(x^4+y^2z^2)+z^2(y^4+z^2x^2)}\right)\end{array}\right)$$

$$\overset{?}{\le}\frac{3}{\sqrt{2}}$$

$$\Leftrightarrow 9(x^4+y^2z^2)(y^4+z^2x^2)(z^4+x^2y^2)$$

$$-2\left(\begin{array}{c}\left(x^2(y^4+z^2x^2)+y^2(z^4+x^2y^2)+z^2(x^4+y^2z^2)\right)\\ *\\ \left(x^2(z^4+x^2y^2)+y^2(x^4+y^2z^2)+z^2(y^4+z^2x^2)\right)\end{array}\right)\overset{?}{\ge}0$$

$$\Leftrightarrow x^2y^2z^2\left(\sum_{cyc}x^6\right)+\sum_{cyc}x^6y^6-6x^4y^4z^4\overset{?}{\ge}0$$

$$\Leftrightarrow x^2y^2z^2\left(3x^2y^2z^2+\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^4-\sum_{cyc}x^2y^2\right)\right)$$

$$+\left(3x^4y^4z^4+\left(\sum_{cyc}x^2y^2\right)\left(\sum_{cyc}x^4y^4-x^2y^2z^2\sum_{cyc}x^2\right)\right)-6x^4y^4z^4\overset{?}{\ge}0$$

$$\Leftrightarrow x^2y^2z^2\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^4-\sum_{cyc}x^2y^2\right)$$

$$+\left(\sum_{cyc} x^2y^2\right)\left(\sum_{cyc} x^4y^4 - x^2y^2z^2\sum_{cyc} x^2\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow x^2y^2z^2\left(\sum_{cyc} x^2\right).\frac{1}{2}\sum_{cyc}\left(x^2-y^2\right)^2 + \left(\sum_{cyc} x^2y^2\right).\frac{1}{2}\sum_{cyc}\left(x^2y^2-y^2z^2\right)^2 \overset{?}{\geq} 0$$

$$\to \text{true} \therefore \sqrt{\frac{2}{1+a^2}} + \sqrt{\frac{2}{1+b^2}} + \sqrt{\frac{2}{1+c^2}} \leq 3\ \forall\, a,b,c \in \mathbb{R} \,\middle|\, abc = 1,$$

$$'' ='' \text{ iff } a = b = c = 1 \text{ (QED)}$$