ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in \mathbb{R}$ and $abc(a+b)(b+c)(c+a) = 8$, then prove that :

$$a^2 + b^2 + c^2 \geq 3$$

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$$a^2 + b^2 + c^2 \geq 3 \Leftrightarrow \left(\sum_{cyc} a^2\right)^3 \geq 27 = \frac{27}{8}.abc(a+b)(b+c)(c+a)$$

$$\Leftrightarrow 8\sum_{cyc} a^6 + 24\sum_{cyc} a^4b^2 + 24\sum_{cyc} a^2b^4 \overset{(*)}{\geq} 27abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) + 6a^2b^2c^2$$

Now, $\forall\, x, y, z \geq 0$, via Schur + A − G, we get : $2\sum_{cyc} x^3 \geq \sum_{cyc} x^2y + \sum_{cyc} xy^2$ and

choosing $x = a^2, y = b^2, z = c^2$, we have : $2\sum_{cyc} a^6 \geq \sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 \to (1)$

Again, $\forall\, \alpha, \beta, \gamma \in \mathbb{R}, \alpha^2 + \beta^2 + \gamma^2 \geq \alpha\beta + \beta\gamma + \gamma\alpha$ and choosing $\alpha = a^2b, \beta = b^2c$, $\gamma = c^2a$, we have : $\sum_{cyc} a^4b^2 \geq abc\sum_{cyc} ab^2 \to (2)$ and choosing

$\alpha = ab^2, \beta = bc^2, \gamma = ca^2$, we have : $\sum_{cyc} a^2b^4 \geq abc\sum_{cyc} a^2b \to (3)$ and also,

$$\sum_{cyc} a^6 - 3a^2b^2c^2 = \left(\sum_{cyc} a^2\right)\left(\frac{1}{2}\sum_{cyc}(a^2 - b^2)^2\right) \geq 0 \Rightarrow \sum_{cyc} a^6 \geq 3a^2b^2c^2 \to (4)$$

$$\text{So, } 8\sum_{cyc} a^6 + 24\sum_{cyc} a^4b^2 + 24\sum_{cyc} a^2b^4 \overset{\text{via (2) and (3)}}{\geq}$$

$$2\sum_{cyc} a^6 + 6\sum_{cyc} a^6 + 24abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) \overset{\text{via (1) and (4)}}{\geq}$$

$$6a^2b^2c^2 + 3\sum_{cyc} a^4b^2 + 3\sum_{cyc} a^2b^4 + 24abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right)$$

$$\overset{\text{via (2) and (3)}}{\geq} 6a^2b^2c^2 + 3abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) + 24abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right)$$

$$= 27abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) + 6a^2b^2c^2 \Rightarrow (*) \text{ is true } \therefore a^2 + b^2 + c^2 \geq 3$$

$\forall\, a, b, c \in \mathbb{R} \mid abc(a+b)(b+c)(c+a) = 8$,

$'' ='' $ iff $(a = b = c = 1)$ or $(a = b = c = -1)$ (QED)