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If $a, \mathrm{b}, \mathrm{c} > 0$ and $a^2 + \mathrm{b}^2 + \mathrm{c}^2 = 1$, then prove that :

$$\frac{1}{1-a} + \frac{1}{1-\mathrm{b}} + \frac{1}{1-\mathrm{c}} \ge \frac{3\sqrt{3}+9}{2}$$

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$$\frac{1}{1-a} + \frac{1}{1-\mathrm{b}} + \frac{1}{1-\mathrm{c}} = \sum_{\mathrm{cyc}} \frac{1+a}{1-a^2} \overset{a^2+\mathrm{b}^2+\mathrm{c}^2=1}{=} \sum_{\mathrm{cyc}} \frac{1+a}{\mathrm{b}^2+\mathrm{c}^2}$$

$$= \sum_{\mathrm{cyc}} \frac{1}{\mathrm{b}^2+\mathrm{c}^2} + \sum_{\mathrm{cyc}} \frac{a}{\mathrm{b}^2+\mathrm{c}^2} \overset{\text{Bergstrom}}{\ge} \frac{9}{2\sum_{\mathrm{cyc}} a^2} + \sum_{\mathrm{cyc}} \frac{a}{\mathrm{b}^2+\mathrm{c}^2} \overset{a^2+\mathrm{b}^2+\mathrm{c}^2=1}{=}$$

$$\frac{9}{2} + \sum_{\mathrm{cyc}} \frac{a}{\mathrm{b}^2+\mathrm{c}^2} \overset{?}{\ge} \frac{3\sqrt{3}+9}{2} \Leftrightarrow \sum_{\mathrm{cyc}} \frac{a}{\mathrm{b}^2+\mathrm{c}^2} \boxed{\underset{(*)}{\overset{?}{\ge}}} \frac{3\sqrt{3}}{2}$$

Assigning $\mathrm{b}+\mathrm{c} = x, \mathrm{c}+a = \mathrm{y}, a+\mathrm{b} = \mathrm{z} \Rightarrow x+\mathrm{y}-\mathrm{z} = 2\mathrm{c} > 0, \mathrm{y}+\mathrm{z}-x = 2a > 0$ and $\mathrm{z}+x-\mathrm{y} = 2\mathrm{b} > 0 \Rightarrow x+\mathrm{y} > \mathrm{z}, \mathrm{y}+\mathrm{z} > x, \mathrm{z}+x > \mathrm{y} \Rightarrow x, \mathrm{y}, \mathrm{z}$ form sides of a triangle with semiperimeter, circumradius and inradius $= \mathrm{s}, \mathrm{R}, \mathrm{r}$ (*say*);

$$\text{so } 2\sum_{\mathrm{cyc}} a = \sum_{\mathrm{cyc}} x = 2\mathrm{s} \Rightarrow \sum_{\mathrm{cyc}} a = \mathrm{s} \to (1) \Rightarrow a = \mathrm{s}-x, \mathrm{b} = \mathrm{s}-\mathrm{y}, \mathrm{c} = \mathrm{s}-\mathrm{z}$$

$$\therefore a\mathrm{bc} = \mathrm{r}^2\mathrm{s} \to (2) \text{ and such substitutions} \Rightarrow \sum_{\mathrm{cyc}} a\mathrm{b} = \sum_{\mathrm{cyc}} (\mathrm{s}-x)(\mathrm{s}-\mathrm{y})$$

$$\Rightarrow \sum_{\mathrm{cyc}} a\mathrm{b} = 4\mathrm{Rr}+\mathrm{r}^2 \to (3), \sum_{\mathrm{cyc}} a^2 = \left(\sum_{\mathrm{cyc}} a\right)^2 - 2\sum_{\mathrm{cyc}} a\mathrm{b} \overset{\text{via (1) and (3)}}{=}$$

$$\mathrm{s}^2 - 2(4\mathrm{Rr}+\mathrm{r}^2) \Rightarrow \sum_{\mathrm{cyc}} a^2 = \mathrm{s}^2 - 8\mathrm{Rr} - 2\mathrm{r}^2 \to (4), \text{and}$$

$$\sum_{\mathrm{cyc}} a^2\mathrm{b}^2 = \left(\sum_{\mathrm{cyc}} a\mathrm{b}\right)^2 - 2a\mathrm{bc}\left(\sum_{\mathrm{cyc}} a\right) \overset{\text{via (1),(2) and (3)}}{=} (4\mathrm{Rr}+\mathrm{r}^2)^2 - 2\mathrm{r}^2\mathrm{s}.\mathrm{s}$$

$$\Rightarrow \sum_{\mathrm{cyc}} a^2\mathrm{b}^2 = \mathrm{r}^2\left((4\mathrm{R}+\mathrm{r})^2 - 2\mathrm{s}^2\right) \to (5)$$

$$\text{Now, } \sum_{\mathrm{cyc}} \frac{a}{\mathrm{b}^2+\mathrm{c}^2} = \sum_{\mathrm{cyc}} \frac{a^4}{a^3\mathrm{b}^2 + a^3\mathrm{c}^2} \overset{\text{Bergstrom}}{\ge} \frac{\left(\sum_{\mathrm{cyc}} a^2\right)^2}{\sum_{\mathrm{cyc}}(a^3\mathrm{b}^2 + a^2\mathrm{b}^3)}$$

$$= \frac{\left(\sum_{\mathrm{cyc}} a^2\right)^2}{\sum_{\mathrm{cyc}}\left(a^2\mathrm{b}^2\left(\sum_{\mathrm{cyc}} a - \mathrm{c}\right)\right)} = \frac{\left(\sum_{\mathrm{cyc}} a^2\right)^2}{\left(\sum_{\mathrm{cyc}} a\right)\left(\sum_{\mathrm{cyc}} a^2\mathrm{b}^2\right) - a\mathrm{bc}\sum_{\mathrm{cyc}} a\mathrm{b}} \overset{?}{\ge} \frac{3\sqrt{3}}{2} \overset{a^2+\mathrm{b}^2+\mathrm{c}^2=1}{=}$$

$$\frac{3\sqrt{3}}{2\sqrt{\sum_{\mathrm{cyc}} a^2}} \Leftrightarrow 4\left(\sum_{\mathrm{cyc}} a^2\right)^5 \ge 27\left(\left(\sum_{\mathrm{cyc}} a\right)\left(\sum_{\mathrm{cyc}} a^2\mathrm{b}^2\right) - a\mathrm{bc}\sum_{\mathrm{cyc}} a\mathrm{b}\right)^2$$

$$\overset{\text{via (1),(2),(3),(4) and (5)}}{\Leftrightarrow}$$

$$4\left(s^2-8Rr-2r^2\right)^5-27\left(sr^2\left((4R+r)^2-2s^2\right)-r^2s\left(4Rr+r^2\right)\right)^2\boxed{\overset{(**)}{\geq}}0\ and\ \because$$

$$P=4\left(s^2-16Rr+5r^2\right)^5+4r(40R-35r)\left(s^2-16Rr+5r^2\right)^4$$
$$+4r^2\left(640R^2-1120Rr+463r^2\right)\left(s^2-16Rr+5r^2\right)^3$$
$$+4r^3\left(5120R^3-13008R^2r+10572Rr^2-3025r^3\right)\left(s^2-16Rr+5r^2\right)^2\overset{Gerretsen}{\geq}0$$

$$\left(\begin{array}{c}\because 5120t^3-13008t^2+10572t-3025\left(t=\dfrac{R}{r}\right)\\ =(t-2)\left(5120t^2-2768t+5036\right)+7047\overset{Euler}{\geq}7047>0\end{array}\right)$$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)\geq P\Leftrightarrow$

$$\left(4688R^4-14680R^3r+18093R^2r^2-10750Rr^3+2495r^4\right)s^2\boxed{\overset{(***)}{\geq}}$$
$$r\left(73728R^5-248832R^4r+337536R^3r^2-232020R^2r^3+77085Rr^4-9117r^5\right)$$
$$\because 4688t^4-14680t^3+18093t^2-10750t+2495$$
$$=(t-2)\left(4688t^3-5304t^2+7485t+4220\right)+10935\overset{Euler}{\geq}10935>0$$
$$\therefore \text{LHS of } (***)\overset{Rouche}{\geq}$$
$$\left(\begin{array}{c}4688R^4-14680R^3r+18093R^2r^2-10750Rr^3\\+2495r^4\end{array}\right)\left(\begin{array}{c}2R^2+10Rr-r^2\\-2(R-2r).\sqrt{R^2-2Rr}\end{array}\right)$$
$$\overset{?}{\geq}\text{RHS of }(***)$$
$$\Leftrightarrow 9376R^6-56208R^5r+133530R^4r^2-163426R^3r^3$$
$$+111417R^2r^4-41385Rr^5+6622r^6$$
$$\overset{?}{\geq}2(R-2r).\sqrt{R^2-2Rr}.\left(\begin{array}{c}4688R^4-14680R^3r+18093R^2r^2-10750Rr^3\\+2495r^4\end{array}\right)\Leftrightarrow$$
$$(R-2r)\left(9376R^5-37456R^4r+58618R^3r^2-46190R^2r^3+19037Rr^4-3311r^5\right)$$
$$\boxed{\underset{(****)}{\overset{?}{\geq}}}2(R-2r).\sqrt{R^2-2Rr}.\left(\begin{array}{c}4688R^4-14680R^3r+18093R^2r^2-10750Rr^3\\+2495r^4\end{array}\right)$$

$$and\ \because 9376t^5-37456t^4+58618t^3-46190t^2+19037t-3311r^5$$
$$=(t-2)\left(24t^4+9352t^3(t-2)+21210t^2-3770t+11497\right)+19683\overset{Euler}{\geq}$$
$$19683>0\ and\ \because R-2r\overset{Euler}{\geq}0\ \therefore \text{in order to prove } (****), \text{it suffices to prove :}$$
$$\left(9376R^5-37456R^4r+58618R^3r^2-46190R^2r^3+19037Rr^4-3311r^5\right)^2$$
$$>4\left(R^2-2Rr\right)\left(4688R^4-14680R^3r+18093R^2r^2-10750Rr^3+2495r^4\right)^2$$
$$\Leftrightarrow 24002560t^9-139530240t^8+351802368t^7-468261888t^6+276650496t^5$$
$$+91963584t^4-285463896t^3+214237449t^2-76262814t+10962721\boxed{\boxed{>}}0$$
$$\Leftrightarrow (t-2)\left((t-2)Q+2554105446\right)+387420489>0, \text{where } Q=$$
$$2242560t^7+21760000(t-2)+81712128t^5+32666624t^4+80468480t^3$$
$$+283171008t^2+525346216t+1182938281\rightarrow \text{true}\because t\overset{Euler}{\geq}2\Rightarrow(****)$$
$$\Rightarrow(***)\Rightarrow(**)\Rightarrow(*)\text{ is true }\therefore \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}\geq\frac{3\sqrt{3}+9}{2},$$
$$\forall\, a,b,c>0\,\middle|\,a^2+b^2+c^2=1,\text{''}=\text{'' iff } a=b=c=\frac{1}{\sqrt{3}}\ \text{(QED)}$$