ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in \mathbb{R}$ and $(a+b+c)(a^2+1)(b^2+1)(c^2+1)=24$, then prove that : $a^2+b^2+c^2 \ge 3$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$24=(a+b+c)(a^2+1)(b^2+1)(c^2+1) \le \sqrt{3\sum_{cyc} a^2}.(a^2+1)(b^2+1)(c^2+1)=$$

$$=\sqrt{3\sum_{cyc} a^2}.\left(a^2b^2c^2+1+\sum_{cyc} a^2+\sum_{cyc} a^2b^2\right)$$

$$\le \sqrt{3\sum_{cyc} a^2}.\left(\frac{1}{27}\left(\sum_{cyc} a^2\right)^3+1+\sum_{cyc} a^2+\frac{1}{3}\left(\sum_{cyc} a^2\right)^2\right)$$

$$=\sqrt{3\sum_{cyc} a^2}.\frac{\left(\sum_{cyc} a^2\right)^3+27+27\sum_{cyc} a^2+9\left(\sum_{cyc} a^2\right)^2}{27}$$

$$\Rightarrow 576 \le \frac{\left(3\sum_{cyc} a^2\right)\left(\left(\sum_{cyc} a^2\right)^3+27+27\sum_{cyc} a^2+9\left(\sum_{cyc} a^2\right)^2\right)^2}{729}$$

$$\Rightarrow t\left(t^3+9t^2+27t+27\right)^2-243.576 \ge 0 \quad \left(t=\sum_{cyc} a^2\right)$$

$$\Rightarrow t^7+18t^6+135t^5+540t^4+1215t^3+1458t^2+729t-139968 \ge 0$$

$$\Rightarrow (t-3)\left(t^6+21t^5+198t^4+1134t^3+4617t^2+15309t+46656\right) \ge 0$$

$$\Rightarrow t \ge 3 \quad \left(\because t\sum_{cyc} a^2 \ge 0\right) \therefore a^2+b^2+c^2 \ge 3$$

$$\forall\, a, b, c \in \mathbb{R} \,\middle|\, (a+b+c)(a^2+1)(b^2+1)(c^2+1)=24,$$

$'' ='' $ iff $a=b=c=1$ (QED)