ROMANIAN MATHEMATICAL MAGAZINE

If $a, b \in \mathbb{R}$ and $ab(a^2+1)(b^2+1) = 4$, then prove that :

$$a^2 + b^2 \geq 2$$

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$$ab(a^2+1)(b^2+1) = 4 \Rightarrow ab = \frac{4}{(a^2+1)(b^2+1)} > 0 \Rightarrow ab > 0 \rightarrow (1)$$

$$\text{Now, } ab(a^2+1)(b^2+1) = 4 \Rightarrow ab(a^2b^2 + 1 + a^2 + b^2) = 4 \Rightarrow a^2 + b^2$$

$$= \frac{4}{ab} - a^2b^2 - 1 \overset{?}{\geq} 2 \Leftrightarrow 4 - x^3 - 3x \overset{?}{\geq} 0 \ \left(\because x = ab > 0 \ldots \text{via } (1)\right)$$

$$\Leftrightarrow x^3 + 3x - 4 \overset{?}{\leq} 0 \Leftrightarrow (x-1)(x^2+x+4) \overset{?}{\leq} 0 \Leftrightarrow x \underbrace{\overset{?}{\leq}}_{(*)} 1 \ (\because x^2 + x + 4 > 0)$$

$$\text{We have : } ab(a^2b^2 + 1 + a^2 + b^2) = 4 \Rightarrow 4 = a^3b^3 + ab + ab(a^2+b^2)$$

$$\overset{A-G}{\geq} a^3b^3 + ab + ab(2ab) \ \left(\because ab > 0 \ldots \text{via } (1)\right) \Rightarrow x^3 + 2x^2 + x - 4 \leq 0$$

$$\Rightarrow (x-1)(x^2+3x+4) \leq 0 \Rightarrow x \leq 1 \ (\because x^2 + 3x + 4 > 0) \Rightarrow (*) \text{ is true}$$

$$\therefore a^2 + b^2 \geq 2 \ \forall \ a, b \in \mathbb{R} \,\middle|\, ab(a^2+1)(b^2+1) = 4,$$

$$'' ='' \ \text{ iff } (a = b = 1) \text{ or } (a = b = -1) \ (\text{QED})$$