ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $a^2+b^2+c^2+a+b+c\ge 6$, then prove that :

$$\frac{a^2}{\sqrt{b+c}}+\frac{b^2}{\sqrt{c+a}}+\frac{c^2}{\sqrt{a+b}}\ge\frac{3}{\sqrt{2}}$$

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$$6\le a^2+b^2+c^2+a+b+c\le\sum_{cyc}a^2+\sqrt{3\sum_{cyc}a^2}=\frac{x^2}{3}+x$$

$$\left(x=\sqrt{3\sum_{cyc}a^2}\right)\Rightarrow x^2+3x-18\ge 0\Rightarrow(x+6)(x-3)\ge 0\Rightarrow x\ge 3\to(1)$$

Now, WLOG assuming $a\ge b\ge c\Rightarrow a^2\ge b^2\ge c^2$ and $\frac{1}{\sqrt{b+c}}\ge\frac{1}{\sqrt{c+a}}\ge\frac{1}{\sqrt{a+b}}$

$$\therefore \text{ via Chebyshev, }\frac{a^2}{\sqrt{b+c}}+\frac{b^2}{\sqrt{c+a}}+\frac{c^2}{\sqrt{a+b}}\ge\frac{1}{3}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{\sqrt{b+c}}\right)$$

$$\overset{\text{Bergstrom}}{\ge}\frac{1}{3}\left(\sum_{cyc}a^2\right)\left(\frac{9}{\sum_{cyc}\sqrt{b+c}}\right)\overset{\text{CBS}}{\ge}\left(3\sum_{cyc}a^2\right)\left(\frac{1}{\sqrt{3}.\sqrt{2\sum_{cyc}a}}\right)$$

$$=\left(3\sum_{cyc}a^2\right)\left(\frac{1}{\sqrt{6}.\sqrt{\sqrt{3\sum_{cyc}a^2}}}\right)=\frac{x^2}{\sqrt{6x}}=\frac{x\sqrt{x}}{\sqrt{6}}\overset{\text{via (1)}}{\ge}\frac{3.\sqrt{3}}{\sqrt{3}.\sqrt{2}}$$

$$\Rightarrow\frac{a^2}{\sqrt{b+c}}+\frac{b^2}{\sqrt{c+a}}+\frac{c^2}{\sqrt{a+b}}\ge\frac{3}{\sqrt{2}}$$

$$\forall\, a,b,c>0\,\big|\,a^2+b^2+c^2+a+b+c\ge 6\ \text{(QED)}$$