ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \in \mathbb{R}$ and $(a^3+1)(b^3+1)(c^3+1)=729$,

then prove that : $a^2+b^2+c^2 \geq 12$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(a^2+2)^4-16(a^3+1)^2=a^8-8a^6+24a^4-32a^3+32a^2$$

$$=a^2(a-2)^2\left(a^2(a+2)^2+8\right)\geq 0\ \forall\, a\in\mathbb{R} \therefore (a^3+1)^2 \overset{(1)}{\leq} \frac{(a^2+2)^4}{16}$$

$$\text{Now, } 729^2=\prod_{\text{cyc}}(a^3+1)^2 \overset{\text{via (1) and analogs}}{\leq} \prod_{\text{cyc}}\frac{(a^2+2)^4}{16}$$

$$\Rightarrow 27\leq \frac{1}{8}\prod_{\text{cyc}}(a^2+2)\ \left(\because \prod_{\text{cyc}}(a^2+2)>0\ \forall\, a,b,c\in\mathbb{R}\right)$$

$$\Rightarrow 216\leq a^2b^2c^2+8+4\sum_{\text{cyc}}a^2+2\sum_{\text{cyc}}a^2b^2$$

$$\leq \frac{1}{27}\left(\sum_{\text{cyc}}a^2\right)^3+8+4\sum_{\text{cyc}}a^2+\frac{2}{3}\left(\sum_{\text{cyc}}a^2\right)^2$$

$$\left(\because \left(\sum_{\text{cyc}}a^2\right)^3\geq 27a^2b^2c^2 \text{ and } \left(\sum_{\text{cyc}}a^2\right)^2\geq 3\sum_{\text{cyc}}a^2b^2\ \forall\, a,b,c\in\mathbb{R}\right)$$

$$=\frac{t^3+216+108t+18t^2}{27}\ \left(t=\sum_{\text{cyc}}a^2\right)\Rightarrow t^3+18t^2+108t-5616\geq 0$$

$$\Rightarrow (t-12)(t^2+30t+48)\geq 0\Rightarrow t\geq 12$$

$$\left(\because t^2+30t+48=\left(\sum_{\text{cyc}}a^2\right)^2+30\left(\sum_{\text{cyc}}a^2\right)+48>0\ \forall\, a,b,c\in\mathbb{R}\right)$$

$$\therefore a^2+b^2+c^2\geq 12\ \forall\, a,b,c\in\mathbb{R}\,\middle|\,(a^3+1)(b^3+1)(c^3+1)=729,$$

$$''='' \text{ iff } a=b=c=2\ \text{(QED)}$$