ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0$ and $abc=1$, then prove that :

$$\frac{a}{\sqrt{1+a}}+\frac{b}{\sqrt{1+b}}+\frac{c}{\sqrt{1+c}}\ge\frac{3\sqrt{2}}{2}$$

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$$\left(\frac{a}{\sqrt{1+a}}+\frac{b}{\sqrt{1+b}}+\frac{c}{\sqrt{1+c}}\right)^2\ge 3\sum_{cyc}\frac{bc}{\sqrt{(1+b)(1+c)}}$$

$$=3\sum_{cyc}\frac{\frac{1}{yz}}{\sqrt{\left(1+\frac{1}{y}\right)\left(1+\frac{1}{z}\right)}}\left(x=\frac{1}{a},y=\frac{1}{b},z=\frac{1}{c}\right)\overset{xyz=1}{=}3\sum_{cyc}\frac{x\sqrt{yz}}{\sqrt{(y+1)(z+1)}}$$

$$\overset{xyz=1}{=}3\sum_{cyc}\frac{x\sqrt{x}}{\sqrt{x^2(y+1)(z+1)}}\overset{Radon}{\ge}3.\frac{\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}\left(x^2(yz+y+z+1)\right)}}\overset{xyz=1}{=}$$

$$3.\frac{\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}x+\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-3+\left(\sum_{cyc}x\right)^2-2\sum_{cyc}xy}}$$

$$=\frac{3\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}x-3+\left(\sum_{cyc}x\right)^2+\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x-2\right)}}$$

$$\ge\frac{3\left(\sum_{cyc}x\right)^{\frac{3}{2}}}{\sqrt{\sum_{cyc}x-3+\left(\sum_{cyc}x\right)^2+\frac{\left(\sum_{cyc}x\right)^2}{3}\left(\sum_{cyc}x-2\right)}}$$

$$\left(\because\sum_{cyc}x\overset{A-G}{\ge}3\sqrt[3]{xyz}\overset{xyz=1}{=}3\Rightarrow\sum_{cyc}x-2\ge1>0\right)\overset{?}{\ge}\frac{9}{2}$$

$$\Leftrightarrow\frac{t^3}{t-3+t^2+\frac{t^2(t-2)}{3}}\overset{?}{\ge}\frac{9}{4}\left(t=\sum_{cyc}x\right)\Leftrightarrow\frac{t^3}{3t-9+3t^2+t^2(t-2)}\overset{?}{\ge}\frac{3}{4}$$

$$\Leftrightarrow t^3-3t^2-9t+27\overset{?}{\ge}0\Leftrightarrow(t-3)^2(t+3)\overset{?}{\ge}0\to\text{true}$$

$$\therefore\left(\frac{a}{\sqrt{1+a}}+\frac{b}{\sqrt{1+b}}+\frac{c}{\sqrt{1+c}}\right)^2\ge\frac{9}{2}\Rightarrow\frac{a}{\sqrt{1+a}}+\frac{b}{\sqrt{1+b}}+\frac{c}{\sqrt{1+c}}\ge\frac{3\sqrt{2}}{2}$$

$\forall a,b,c>0\,|\,abc=1,''='' \text{ iff } a=b=c=1$ (QED)