ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, then prove that :

$$\frac{a^4}{a^3+2b^3}+\frac{b^4}{b^3+2c^3}+\frac{c^4}{c^3+2a^3}\ge\frac{a+b+c}{3}$$

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$$\text{LHS}-\text{RHS}=\sum_{\text{cyc}}\left(\frac{a^4}{a^3+2b^3}-\frac{a}{3}\right)=\sum_{\text{cyc}}\frac{2a(a^3-b^3)}{3(a^3+2b^3)}\overset{?}{\ge}0$$

$$\Leftrightarrow\frac{a(a^3+2b^3-3b^3)}{a^3+2b^3}\overset{?}{\ge}0\Leftrightarrow\frac{1}{3}\sum_{\text{cyc}}a\underset{(*)}{\overset{?}{\ge}}\sum_{\text{cyc}}\frac{ab^3}{a^3+2b^3}$$

$$\text{Now, }\sum_{\text{cyc}}\frac{ab^3}{a^3+2b^3}\le\sum_{\text{cyc}}\frac{ab^3}{ab(a+b)+b^3}=\sum_{\text{cyc}}\frac{ab^2}{a^2+ab+b^2}\overset{A-G}{\le}\sum_{\text{cyc}}\frac{b.\frac{(a+b)^2}{4}}{\frac{3}{4}.(a+b)^2}$$

$$\left(\because a^2+ab+b^2=\frac{3}{4}.(a+b)^2+\frac{1}{4}.(a-b)^2\right)=\frac{1}{3}\sum_{\text{cyc}}a\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{a^4}{a^3+2b^3}+\frac{b^4}{b^3+2c^3}+\frac{c^4}{c^3+2a^3}\ge\frac{a+b+c}{3}\ \forall a,b,c>0,$$

$$''='' \text{ iff } a=b=c\ (\text{QED})$$