ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, x + y + z = xyz$ then:

$$\sqrt{x^2yz + yz} + \sqrt{y^2xz + xz} + \sqrt{z^2xy + xy} \le 2xyz$$

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Solution by proposer

$$\sqrt{x^2yz + yz} = \sqrt{x^2 - x^2 + x^2yz + yz} = \sqrt{x^2 + x(xyz - x) + yz} =$$
$$= \sqrt{x^2 + x(y + z) + yz} = \sqrt{(x + y)(x + z)}$$

İt can also be shown that $\sqrt{y^2xz + xz} = \sqrt{(x + y)(y + z)}$ and $\sqrt{z^2xy + xy} = \sqrt{(x + z)(y + z)}$ is true

$$\sqrt{(x + y)(x + z)} \overset{AM \ge GM}{\le} \frac{x+y+x+z}{2} = \frac{2x+y+z}{2} \quad (1)$$
$$\Rightarrow \sqrt{(x + y)(y + z)} \le \frac{2y+x+z}{2} \quad (2) \overset{AM \ge GM}{\le}$$
$$\sqrt{(x + z)(y + z)} \le \frac{2z+x+y}{2} \quad (3)$$

$$(1) + (2) + (3) \Rightarrow \sqrt{x^2yz + yz} + \sqrt{y^2xz + xz} + \sqrt{z^2xy + xy} \le \frac{4(x+y+z)}{2} = \frac{4xyz}{2} = 2xyz$$