ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} \in \mathbb{R}^{+}$ such that : $a\mathrm{bc} = 1$, then prove that :

$$\frac{1+2a\mathrm{b}}{1+a\mathrm{c}+\mathrm{bc}}+\frac{1+2\mathrm{bc}}{1+a\mathrm{c}+a\mathrm{b}}+\frac{1+2a\mathrm{c}}{1+a\mathrm{b}+\mathrm{bc}} \geq 3$$

Proposed by Samed Ahmedov and Xumar Mammadli-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{1+2a\mathrm{b}}{1+a\mathrm{c}+\mathrm{bc}}+\frac{1+2\mathrm{bc}}{1+a\mathrm{c}+a\mathrm{b}}+\frac{1+2a\mathrm{c}}{1+a\mathrm{b}+\mathrm{bc}}$$

$$=\frac{1}{1+a\mathrm{c}+\mathrm{bc}}+\frac{1}{1+a\mathrm{c}+a\mathrm{b}}+\frac{1}{1+a\mathrm{b}+\mathrm{bc}}+$$

$$\frac{2a^2\mathrm{b}^2}{a\mathrm{b}+a^2\mathrm{bc}+a\mathrm{b}^2\mathrm{c}}+\frac{2\mathrm{b}^2\mathrm{c}^2}{\mathrm{bc}+a\mathrm{bc}^2+a\mathrm{b}^2\mathrm{c}}+\frac{2a^2\mathrm{c}^2}{a\mathrm{c}+a^2\mathrm{bc}+a\mathrm{bc}^2}$$

$$\overset{\text{Bergstrom and } \because\, abc=1}{\geq} \frac{9}{3+2\sum_{\text{cyc}}a\mathrm{b}}+\frac{2\left(\sum_{\text{cyc}}a\mathrm{b}\right)^2}{\sum_{\text{cyc}}a\mathrm{b}+2\sum_{\text{cyc}}a}$$

$$\geq \frac{9}{3+2\sum_{\text{cyc}}a\mathrm{b}}+\frac{2\left(\sum_{\text{cyc}}a\mathrm{b}\right)^2}{\sum_{\text{cyc}}a\mathrm{b}+\frac{2}{3}\left(\sum_{\text{cyc}}a\mathrm{b}\right)^2}$$

$$\left(\because \left(\sum_{\text{cyc}}a\mathrm{b}\right)^2 \geq 3a\mathrm{bc}\sum_{\text{cyc}}a \overset{abc=1}{=} 3\sum_{\text{cyc}}a \Rightarrow \sum_{\text{cyc}}a \leq \frac{1}{3}\left(\sum_{\text{cyc}}a\mathrm{b}\right)^2\right)$$

$$=\frac{9}{3+2\sum_{\text{cyc}}a\mathrm{b}}+\frac{6\sum_{\text{cyc}}a\mathrm{b}}{3+2\sum_{\text{cyc}}a\mathrm{b}}=\frac{3\left(3+2\sum_{\text{cyc}}a\mathrm{b}\right)}{3+2\sum_{\text{cyc}}a\mathrm{b}}=3$$

$$\therefore \frac{1+2a\mathrm{b}}{1+a\mathrm{c}+\mathrm{bc}}+\frac{1+2\mathrm{bc}}{1+a\mathrm{c}+a\mathrm{b}}+\frac{1+2a\mathrm{c}}{1+a\mathrm{b}+\mathrm{bc}} \geq 3\ \forall\, a,\mathrm{b},\mathrm{c} \in \mathbb{R}^+ \,\middle|\, a\mathrm{bc}=1,$$

$''=''$ iff $a = \mathrm{b} = \mathrm{c} = 1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

$$\frac{1+2ab}{1+bc+ca}+\frac{1+2bc}{1+ca+ab}+\frac{1+2ca}{1+ab+bc}$$

$$=\left(\frac{3+2(ab+bc+ca)}{1+bc+ca}-2\right)+\left(\frac{3+2(ab+bc+ca)}{1+ca+ab}-2\right)+\left(\frac{3+2(ab+bc+ca)}{1+ab+bc}-2\right)=$$

$$=[3+2(ab+bc+ca)]\left(\frac{1}{1+bc+ca}+\frac{1}{1+ca+ab}+\frac{1}{1+ab+bc}\right)-6$$

$$\overset{CBS}{\geq} [3+2(ab+bc+ca)].\frac{9}{3+2(ab+bc+ca)}-6=3,$$

as desired. Equality holds iff $a = b = c = 1$.