ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ with $p \leq 3$, then :

$$\frac{9}{8A}\left(\frac{1}{p}+\frac{3}{q}\right) \geq \sum_{cyc} \frac{1}{a+b} \geq \frac{9}{8A}\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right),$$

where : $A = 1 + \frac{1}{8pq}\sum_{cyc} c(a-b)^2$, $p = a+b+c, q = ab+bc+ca$

When does equality holds ?

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Solution 1 by Soumava Chakraborty-Kolkata-India

Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0, y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y \Rightarrow x, y, z$ form sides of a triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say) yielding $2\sum_{cyc} a = \sum_{cyc} x = 2s \Rightarrow \sum_{cyc} a = s \rightarrow (1) \Rightarrow a = s-x, b = s-y, c = s-z$

$\therefore abc = r^2 s \rightarrow (2)$ and such substitutions $\Rightarrow \sum_{cyc} ab = \sum_{cyc}(s-x)(s-y)$

$\Rightarrow \sum_{cyc} ab = 4Rr + r^2 \rightarrow (3)$ and we have : $A = 1 + \frac{1}{8pq}\cdot\sum_{cyc} c(a-b)^2$

$$= 1 + \frac{1}{8\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)}\cdot\sum_{cyc}\left(c(a^2+b^2-2ab)\right)$$

$$= 1 + \frac{1}{8\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)}\cdot\left(\sum_{cyc}\left(ab\left(\sum_{cyc} a - c\right)\right) - 6abc\right)$$

$$= 1 + \frac{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 9abc}{8\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)} \overset{\text{via (1),(2) and (3)}}{=} 1 + \frac{s(4Rr+r^2) - 9r^2 s}{8s(4Rr+r^2)}$$

$$\Rightarrow A = \frac{9R}{2(4R+r)} \rightarrow (4) \text{ and } \frac{9}{8A}\left(\frac{1}{p}+\frac{3}{q}\right) \overset{3 \geq p}{\geq} \frac{9}{8A}\left(\frac{1}{p}+\frac{p}{q}\right) = \frac{9}{8A}\left(\frac{1}{\sum_{cyc} a}+\frac{\sum_{cyc} a}{\sum_{cyc} ab}\right)$$

$$\overset{\text{via (1),(3) and (4)}}{=} \frac{9}{8}\cdot\frac{2(4R+r)}{9R}\cdot\left(\frac{1}{s}+\frac{s}{4Rr+r^2}\right) = \frac{9}{8}\cdot\frac{2(4R+r)}{9R}\cdot\frac{s^2+4Rr+r^2}{rs(4R+r)}$$

$$= \frac{s^2+4Rr+r^2}{4Rrs} = \frac{\sum_{cyc} xy}{xyz} = \sum_{cyc}\frac{1}{x} = \sum_{cyc}\frac{1}{a+b} \Rightarrow \boxed{\frac{9}{8A}\left(\frac{1}{p}+\frac{3}{q}\right) \geq \sum_{cyc}\frac{1}{a+b}}$$

$$\text{Again, } \frac{9}{8A}\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right) = \frac{9}{8A}\left(\frac{4}{3}+\frac{p}{q}+\frac{q}{p}-2\right) \leq \frac{9}{8A}\left(\frac{4}{3}+\frac{p}{q}+1-2\right)$$

$$\left(\because p\le 3\le\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}ab}=\frac{p^2}{q}\Rightarrow\frac{q}{p}\le 1\right)=\frac{9}{8A}\left(\frac{p}{q}+\frac{1}{3}\right)\overset{\frac{1}{3}\le\frac{1}{p}}{\le}\frac{9}{8A}\left(\frac{p}{q}+\frac{1}{p}\right)$$

$$=\frac{9}{8A}\left(\frac{1}{\sum_{cyc}a}+\frac{\sum_{cyc}a}{\sum_{cyc}ab}\right)\overset{\text{via (1),(3) and (4)}}{=}\frac{9}{8}\cdot\frac{2(4R+r)}{9R}\cdot\left(\frac{1}{s}+\frac{s}{4Rr+r^2}\right)$$

$$=\frac{9}{8}\cdot\frac{2(4R+r)}{9R}\cdot\frac{s^2+4Rr+r^2}{rs(4R+r)}=\frac{s^2+4Rr+r^2}{4Rrs}=\frac{\sum_{cyc}xy}{xyz}=\sum_{cyc}\frac{1}{x}=\sum_{cyc}\frac{1}{a+b}$$

$$\therefore\boxed{\sum_{cyc}\frac{1}{a+b}\ge\frac{9}{8A}\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right)}$$

$$\therefore\frac{9}{8A}\left(\frac{1}{p}+\frac{3}{q}\right)\ge\sum_{cyc}\frac{1}{a+b}\ge\frac{9}{8A}\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right)\ \forall\, a,b,c>0 \text{ with}$$

$p\le 3$ where $p=a+b+c, q=ab+bc+ca$, " = " iff $a=b=c=1$ (QED)

Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

We have $\sum_{cyc}\frac{1}{a+b}=\frac{p^2+q}{(a+b)(b+c)(c+a)}$ and

$A=\frac{9(a+b)(b+c)(c+a)}{8pq}$, then the problem becomes to prove that

$$pq\left(\frac{1}{p}+\frac{3}{q}\right)\ge p^2+q\ge pq\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right),$$

We have $pq\left(\frac{1}{p}+\frac{3}{q}\right)=q+3p\ge q+p^2$ and

$$pq\left(\frac{4}{3}+\left(\sqrt{\frac{p}{q}}-\sqrt{\frac{q}{p}}\right)^2\right)=p^2+q^2-\frac{2pq}{3}\overset{?}{\le}p^2+q$$

$$\Leftrightarrow 3+2p\ge 3q\ \Leftrightarrow\ (3-p)(1+p)+(p^2-3q)\ge 0,$$

which is true because $p\le 3$ and $p^2\ge 3q$. So the proof is complete.

Equality holds iff $a=b=c=1$.