ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c \geq 0$ then :

$$\frac{a^3}{(a+1)(b+1)}+\frac{b^3}{(b+1)(c+1)}+\frac{c^3}{(c+1)(a+1)} \geq \frac{3(a+b+c)(a^2+b^2+c^2)}{(a+b+c+3)^2}$$

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Let $p := a+b+c$ and $q := a^2+b^2+c^2$. We have $3q \geq p^2$.

By AM − GM inequality, we have

$$3(a^2b+b^2c+c^2a) \leq a^2b+b^2c+c^2a+(a^3+ab^2)+(b^3+bc^2)+(c^3+ca^2)=pq.$$

By CBS inequality, we have

$$\frac{a^3}{(a+1)(b+1)}+\frac{b^3}{(b+1)(c+1)}+\frac{c^3}{(c+1)(a+1)} \geq \frac{(a^2+b^2+c^2)^2}{\sum_{cyc} a(a+1)(b+1)}$$

$$=\frac{q^2}{\sum_{cyc} a^2b+\sum_{cyc} a^2+\sum_{cyc} ab+\sum_{cyc} a} \geq \frac{q^2}{\frac{pq}{3}+\frac{p^2+q}{2}+p} \overset{?}{\geq} \frac{3pq}{(p+3)^2}.$$

$$\Leftrightarrow 2q(p+3)^2 \geq p(2pq+3p^2+3q+6p) \Leftrightarrow 3pq+6q \geq p^3+2p^2$$
$$\Leftrightarrow (3q-p^2)(p+2) \geq 0,$$

which is true and the proof is complete. Equality holds iff $a=b=c$.