ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$, then :

$$\mathrm{B} + 2 + \sqrt{2} \geq \frac{a}{\mathrm{b}} + \frac{\mathrm{b}}{a} + \frac{a+\mathrm{b}}{\sqrt{a^2+\mathrm{b}^2}} \geq 2 + \sqrt{2} + \mathrm{A}, \text{where :}$$

$$\mathrm{A} = \frac{1}{2\sqrt{2}} \cdot \frac{(a-\mathrm{b})^4}{(a^2+\mathrm{b}^2)^2} + \left(1 + \frac{1}{2\sqrt{2}}\right) \cdot \frac{(a-\mathrm{b})^2}{a^2+\mathrm{b}^2} \text{ and } \mathrm{B} = \frac{1}{2\sqrt{2}} \cdot \frac{(a-\mathrm{b})^4}{a\mathrm{b}(a+\mathrm{b})^2} + \frac{(a-\mathrm{b})^2}{a\mathrm{b}}$$

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$$\frac{a}{\mathrm{b}} + \frac{\mathrm{b}}{a} - 2 - \mathrm{A}$$

$$= \frac{(a-\mathrm{b})^2}{a\mathrm{b}} - \left(1 + \frac{1}{2\sqrt{2}} - 1\right) \cdot \frac{(a-\mathrm{b})^4}{(a^2+\mathrm{b}^2)^2} - \left(2 - \left(1 - \frac{1}{2\sqrt{2}}\right)\right) \cdot \frac{(a-\mathrm{b})^2}{a^2+\mathrm{b}^2}$$

$$= \frac{(a-\mathrm{b})^2}{a\mathrm{b}} - \frac{2(a-\mathrm{b})^2}{a^2+\mathrm{b}^2} - \frac{(a-\mathrm{b})^4}{(a^2+\mathrm{b}^2)^2} + \left(1 - \frac{1}{2\sqrt{2}}\right) \cdot \left(\frac{(a-\mathrm{b})^4}{(a^2+\mathrm{b}^2)^2} + \frac{(a-\mathrm{b})^2}{a^2+\mathrm{b}^2}\right)$$

$$= (a-\mathrm{b})^2 \left(\frac{1}{a\mathrm{b}} - \frac{2}{a^2+\mathrm{b}^2}\right) - \frac{(a-\mathrm{b})^4}{(a^2+\mathrm{b}^2)^2} + \frac{\left(1 - \frac{1}{2\sqrt{2}}\right)(a-\mathrm{b})^2}{a^2+\mathrm{b}^2} \cdot \left(\frac{(a-\mathrm{b})^2}{a^2+\mathrm{b}^2} + 1\right)$$

$$= (a-\mathrm{b})^2 \cdot \frac{(a-\mathrm{b})^2}{a\mathrm{b}(a^2+\mathrm{b}^2)} - \frac{(a-\mathrm{b})^4}{(a^2+\mathrm{b}^2)^2} + \frac{2\left(1 - \frac{1}{2\sqrt{2}}\right)(a-\mathrm{b})^2}{a^2+\mathrm{b}^2} \cdot \frac{a^2+\mathrm{b}^2-a\mathrm{b}}{a^2+\mathrm{b}^2}$$

$$= \frac{(a-\mathrm{b})^4(a^2+\mathrm{b}^2-a\mathrm{b})}{a\mathrm{b}(a^2+\mathrm{b}^2)^2} + \frac{\left(2 - \frac{1}{\sqrt{2}}\right)(a-\mathrm{b})^2(a^2+\mathrm{b}^2-a\mathrm{b})}{(a^2+\mathrm{b}^2)^2}$$

$$= \frac{(a-\mathrm{b})^2(a^2+\mathrm{b}^2-a\mathrm{b})}{(a^2+\mathrm{b}^2)^2} \cdot \left(\frac{(a-\mathrm{b})^2}{a\mathrm{b}} + 2 - \frac{1}{\sqrt{2}}\right)$$

$$\therefore \frac{a}{\mathrm{b}} + \frac{\mathrm{b}}{a} - 2 - \mathrm{A} \overset{(*)}{\geq} \frac{(a-\mathrm{b})^2(a^2+\mathrm{b}^2-a\mathrm{b})}{(a^2+\mathrm{b}^2)^2} \cdot \left(\frac{a^2+\mathrm{b}^2}{a\mathrm{b}} - \frac{1}{\sqrt{2}}\right)$$

$$\text{Again, } \sqrt{2} - \frac{a+\mathrm{b}}{\sqrt{a^2+\mathrm{b}^2}} = \frac{\left(\sqrt{2(a^2+\mathrm{b}^2)} - (a+\mathrm{b})\right)\left(\sqrt{2(a^2+\mathrm{b}^2)} + (a+\mathrm{b})\right)}{\sqrt{a^2+\mathrm{b}^2} \cdot \left(\sqrt{2(a^2+\mathrm{b}^2)} + (a+\mathrm{b})\right)}$$

$$= \frac{2(a^2+\mathrm{b}^2) - (a+\mathrm{b})^2}{\sqrt{2} \cdot (a^2+\mathrm{b}^2) + \sqrt{a^2+\mathrm{b}^2} \cdot (a+\mathrm{b})} = \frac{(a-\mathrm{b})^2}{\sqrt{2} \cdot (a^2+\mathrm{b}^2) + \sqrt{a^2+\mathrm{b}^2} \cdot (a+\mathrm{b})}$$

$$\leq \frac{(a-\mathrm{b})^2}{\sqrt{2} \cdot (a^2+\mathrm{b}^2) + \frac{(a+\mathrm{b})^2}{\sqrt{2}}} \therefore \sqrt{2} - \frac{a+\mathrm{b}}{\sqrt{a^2+\mathrm{b}^2}} \overset{(**)}{\leq} \frac{(a-\mathrm{b})^2}{\sqrt{2} \cdot (a^2+\mathrm{b}^2) + \frac{(a+\mathrm{b})^2}{\sqrt{2}}} \therefore (*), (**) \Rightarrow$$

$$\text{in order to prove : } \frac{a}{\mathrm{b}} + \frac{\mathrm{b}}{a} - 2 - \mathrm{A} \geq \sqrt{2} - \frac{a+\mathrm{b}}{\sqrt{a^2+\mathrm{b}^2}}, \text{it suffices to prove :}$$

$$\frac{(a-\mathrm{b})^2(a^2+\mathrm{b}^2-a\mathrm{b})}{(a^2+\mathrm{b}^2)^2} \cdot \left(\frac{a^2+\mathrm{b}^2}{a\mathrm{b}} - \frac{1}{\sqrt{2}}\right) \geq \frac{(a-\mathrm{b})^2}{\sqrt{2} \cdot (a^2+\mathrm{b}^2) + \frac{(a+\mathrm{b})^2}{\sqrt{2}}}$$

$$\Leftrightarrow \frac{(a-b)^2(a^2+b^2-ab)}{(a^2+b^2)^2}\cdot\left(\sqrt{2}.\frac{a^2+b^2}{ab}-1\right) \geq \frac{\sqrt{2}.(a-b)^2}{\sqrt{2}.(a^2+b^2)+\frac{(a+b)^2}{\sqrt{2}}}$$

$$\Leftrightarrow \boxed{\frac{(a-b)^2(a^2+b^2-ab)}{(a^2+b^2)^2}\cdot\left(\sqrt{2}.\frac{a^2+b^2}{ab}-1\right) \overset{(\bullet)}{\geq} \frac{2(a-b)^2}{2(a^2+b^2)+(a+b)^2}} \text{ and}$$

$\because (a-b)^2 \geq 0$ *and* $\sqrt{2} > 1$, **in order to prove** $(\bullet)$, **it suffices to prove :**

$$\frac{(a^2+b^2-ab)\left(\frac{a^2+b^2}{ab}-1\right)}{(a^2+b^2)^2} \geq \frac{2}{2(a^2+b^2)+(a+b)^2}$$

$$\Leftrightarrow \left(2(a^2+b^2)+(a+b)^2\right)(a^2+b^2-ab)^2 \geq 2ab(a^2+b^2)^2$$

$$\Leftrightarrow 3t^6-6t^5+8t^4-10t^3+8t^2-6t+3 \geq 0\ \left(t=\frac{a}{b}\right)$$

$$\Leftrightarrow \boxed{(3t^4+5t^2+3)(t-1)^2 \geq 0} \to \text{true} \Rightarrow (\bullet) \text{ is true } \therefore \frac{a}{b}+\frac{b}{a}-2-A$$

$$\geq \sqrt{2}-\frac{a+b}{\sqrt{a^2+b^2}} \Rightarrow \boxed{\boxed{\frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}} \geq 2+\sqrt{2}+A}}, ''='' \text{ iff } a=b$$

Now, $B+2+\sqrt{2} \geq \frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}} \Leftrightarrow \frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{(a-b)^2}{ab}+2+\sqrt{2}$

$$\geq \frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}} \Leftrightarrow \frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{a}{b}+\frac{b}{a}+\left(\sqrt{2}-\frac{a+b}{\sqrt{a^2+b^2}}\right) \geq \frac{a}{b}+\frac{b}{a}$$

$$\Leftrightarrow \frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{\left(\sqrt{2(a^2+b^2)}-(a+b)\right)\left(\sqrt{2(a^2+b^2)}+(a+b)\right)}{\sqrt{a^2+b^2}.\left(\sqrt{2(a^2+b^2)}+(a+b)\right)} \geq 0$$

$$\Leftrightarrow \boxed{\frac{1}{2\sqrt{2}}\cdot\frac{(a-b)^4}{ab(a+b)^2}+\frac{(a-b)^2}{\sqrt{a^2+b^2}.\left(\sqrt{2(a^2+b^2)}+(a+b)\right)} \geq 0} \to \text{true} \because a,b>0$$

$$\therefore \boxed{\boxed{B+2+\sqrt{2} \geq \frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}}}}, ''='' \text{ iff } a=b$$

$$\therefore B+2+\sqrt{2} \geq \frac{a}{b}+\frac{b}{a}+\frac{a+b}{\sqrt{a^2+b^2}} \geq 2+\sqrt{2}+A\ \forall\, a,b>0, ''='' \text{ iff } a=b \text{ (QED)}$$