ROMANIAN MATHEMATICAL MAGAZINE

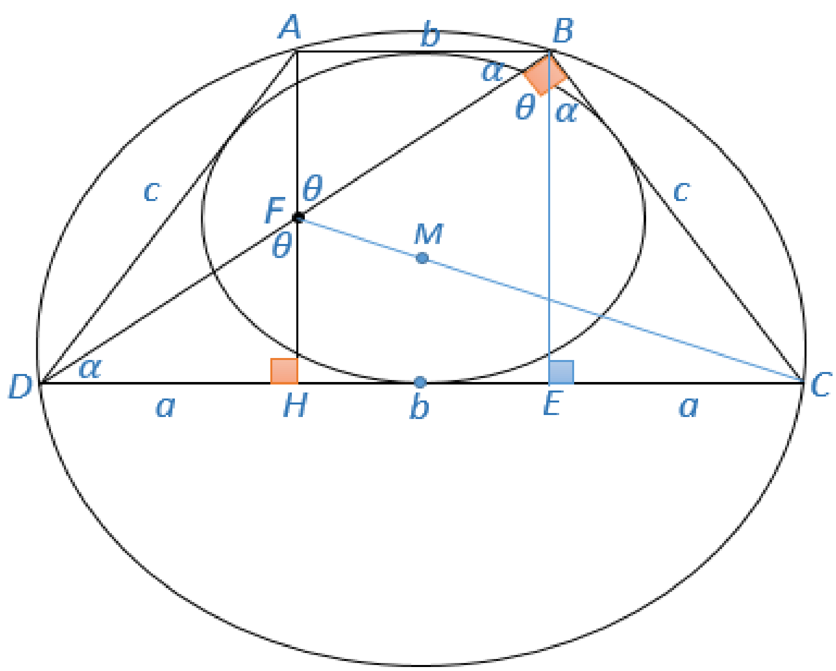
Prove that:

$$\frac{\overline{DB}}{\overline{AH}} = \frac{\overline{FH}}{\overline{FA}} = \varphi \text{ and } R^2 = r^2\varphi^3$$

Proposed by Jafar Nikpour-Iran

Solution by Mirsadix Muzefferov-Azerbaijan

Trapezoid



$$Let\ AD = c, AB = b, DH = a. Then\ BC = c, EC = a, HE = b.$$

$$Also, 2a + 2b = 2c \Rightarrow a + b = c$$

$$In\ \Delta CBD\ (B = 90^\circ)\ BE^2 = AH^2 = DE \cdot EC = (a+b)a\ (1)$$

$$In\ \Delta BEC\ (E = 90^\circ)\ AH^2 = BE^2 = BC^2 - CE^2 = c^2 - a^2 = (a+b)^2 - a^2 = b^2 + 2ab\ (2)$$

$$From\ (1)\ and\ (2)\ we\ have\ (a+b)a = b^2 + 2ab \Rightarrow a^2 - ab - b^2 = 0 \Rightarrow \left(\frac{a}{b}\right)^2 - \frac{a}{b} - 1 = 0$$

$$Let\ \frac{a}{b} = t, t^2 - t - 1 = 0 \Rightarrow t = \frac{1+\sqrt{5}}{2} = \varphi \Rightarrow \frac{a}{b} = \varphi$$

$$\Delta FHC \equiv \Delta FCB \ (FC \ common \ side, HC = CB \ and \ F\hat{H}C = F\hat{B}C = 90^\circ)$$

$$Therefore \ FB = FH, \Delta DBE \sim \Delta FAB \Rightarrow \frac{DB}{BE} = \frac{FB}{AF} \Rightarrow \frac{DB}{AH} = \frac{FH}{AF}$$

$$\Delta DFH \sim \Delta ABF \Rightarrow \frac{FH}{AF} = \frac{a}{b} = \varphi$$

$$(2) \Rightarrow 4r^2 = (2a+b)^2 \Rightarrow \frac{R^2}{r^2} = \frac{2a+b}{b} = 2\varphi + 1$$

$$As \ a \ known \ fact \ \varphi^2 - \varphi - 1 = 0 \Rightarrow \varphi^2 = \varphi + 1, \varphi^3 = \varphi^2 \cdot \varphi = (\varphi+1)\varphi = \varphi^2 + \varphi =$$
$$= \varphi + 1 + \varphi = 2\varphi + 1 \ \ proved$$