ROMANIAN MATHEMATICAL MAGAZINE

Let O, G −be the Toricelli's point and the centroid in ΔABC. Prove that:

$$OG^2 = \frac{a^2+b^2+c^2-4\sqrt{3}F}{18}$$

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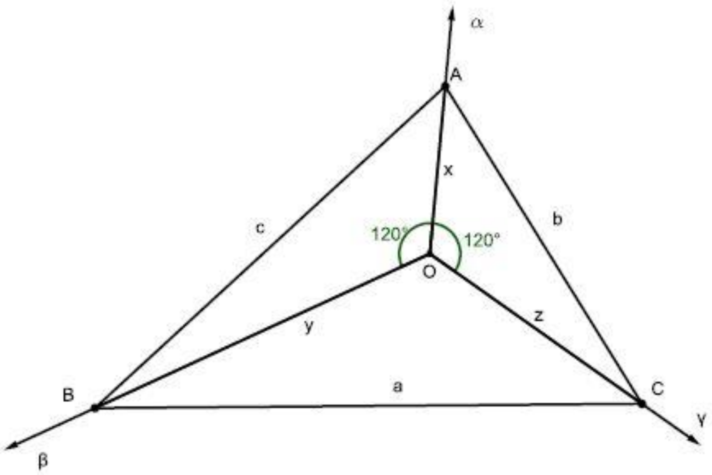

O −Toricelli's point

$$OA = x, OB = y, OC = z$$
$$\sphericalangle AOB = \sphericalangle BOC = \sphericalangle COA = 120°$$
$$[AOB]+[BOC]+[COA] = \sum_{cyc}\frac{1}{2}\cdot xysin120°$$
$$F = \frac{1}{2}sin(180°-60°)\sum_{cyc}xy$$
$$\sum_{cyc}xy = \frac{2F}{sin60°} = \frac{4F}{\sqrt{3}} = \frac{4\sqrt{3}F}{3}$$
$$\sum_{cyc}c^2 = \sum_{cyc}(x^2+y^2-2xycos120°)$$
$$a^2+b^2+c^2 = 2\sum_{cyc}x^2 + 2cos60°\sum_{cyc}xy$$
$$2\sum_{cyc}x^2 = a^2+b^2+c^2 - 2\cdot\frac{1}{2}\cdot\frac{4\sqrt{3}F}{3}$$
$$\sum_{cyc}OA^2 = \frac{1}{2}(a^2+b^2+c^2) - \frac{2\sqrt{3}F}{3}$$

$$3OG^2+\sum_{cyc}GA^2=\frac{1}{2}(a^2+b^2+c^2)-\frac{2\sqrt{3}F}{3}$$

$$3OG^2=\frac{1}{2}(a^2+b^2+c^2)-\frac{2\sqrt{3}F}{3}-\sum_{cyc}\frac{4}{9}m_a^2$$

$$3OG^2=\frac{1}{2}(a^2+b^2+c^2)-\frac{2\sqrt{3}F}{3}-\frac{4}{9}\cdot\frac{3}{4}(a^2+b^2+c^2)$$

$$OG^2=\frac{1}{6}(a^2+b^2+c^2)-\frac{1}{9}(a^2+b^2+c^2)-\frac{2\sqrt{3}F}{9}$$

$$OG^2=\frac{a^2+b^2+c^2-4\sqrt{3}F}{18}$$

Observation: This is a new proof for IONESCU-WEITZENBOCK'S inequality:

$$0\le OG^2=\frac{a^2+b^2+c^2-4\sqrt{3}F}{18}\Rightarrow 0\le a^2+b^2+c^2-4\sqrt{3}F\Rightarrow$$

$$a^2+b^2+c^2\ge 4\sqrt{3}F$$