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In ΔABC the following relationship holds:

$$\frac{1-\cos(B-C)}{h_a}+\frac{1-\cos(C-A)}{h_b}+\frac{1-\cos(A-B)}{h_c}=\frac{R-2r}{Rr}$$

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Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{1-\cos(B-C)}{h_a}=\sum_{cyc}\frac{1}{h_a}-\sum_{cyc}\frac{\cos(B-C)}{h_a}=$$

$$=\sum_{cyc}\frac{1}{\frac{2F}{a}}-\sum_{cyc}\frac{\cos(B-C)}{\frac{2F}{a}}=\frac{1}{2F}\sum_{cyc}a-\frac{1}{2F}\sum_{cyc}a\cos(B-C)=$$

$$=\frac{2s}{2F}-\frac{1}{2F}\sum_{cyc}2R\sin A\cos(B-C)=\frac{s}{F}-\frac{R}{F}\sum_{cyc}\sin A\cos(B-C)=$$

$$=\frac{s}{rs}-\frac{R}{2F}\sum_{cyc}(\sin(A+B-C)+\sin(A-B+C))=$$

$$=\frac{1}{r}-\frac{R}{2F}\sum_{cyc}\big(\sin(\pi-2C)+\sin(\pi-2B)\big)=$$

$$=\frac{1}{r}-\frac{R}{2F}\sum_{cyc}(\sin 2C+\sin 2B)=\frac{1}{r}-\frac{R}{F}\sum_{cyc}\sin 2A=$$

$$=\frac{1}{r}-\frac{R}{F}\cdot 4\sin A\sin B\sin C=\frac{1}{r}-\frac{R}{F}\cdot 4\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}=$$

$$=\frac{1}{r}-\frac{abc}{F\cdot 2R^2}=\frac{1}{r}-\frac{4RF}{F\cdot 2R^2}=\frac{1}{r}-\frac{2}{R}=\frac{R-2r}{Rr}$$