ROMANIAN MATHEMATICAL MAGAZINE

If ΔABC is right in A then:

$$\sqrt[4]{\frac{(-a^6+b^6+c^6)(-a^{14}+b^{14}+c^{14})}{21(a^4-b^2c^2)^2}}=4Rr(2R+r)$$

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$$\Delta ABC \text{ is right at } A\text{, so } a^2=b^2+c^2$$

$$2R=a \text{ and for right angle triangle inradius } r=\frac{1}{2}(b+c-a) \text{ or}$$

$$b+c=2r+a=2r+2R \text{ and}$$

$$2s=a+b+c=2R+2r+2R=4R+2r \text{ or } s=2R+r \ (1)$$

$$\text{Let } a^2=x, b^2=y, c^2=z \text{ then } (-a^6+b^6+c^6)=(-x^3+y^3+z^3)$$

$$=-\left(x^3-(y^3+z^3)\right)\overset{a^2=b^2+c^2 \text{ or } x=y+z}{=}-((y+z)^3-y^3-z^3)$$

$$=-(y^3+z^3+3yz(y+z)-y^3-z^3)=-3yz(y+z) \ (2)$$

$$-a^{14}+b^{14}+c^{14}=-(x^7-y^7-z^7)\overset{a^2=b^2+c^2 \text{ or } x=y+z}{=}-((y+z)^7-y^7-z^7)=$$

$$=-\left(y^7+7y^6z+21y^5z^2+35y^4z^3+35y^3z^4+21y^2z^5+7yz^6+z^7-y^7-z^7\right)=$$

$$=-7yz(y+z)(y^4+2y^3z+3y^2z^2.+2yz^3+z^4 \ (3)$$

$$(a^4-b^2c^2)^2=(x^2-yz)^2\overset{a^2=b^2+c^2 \text{ or } x=y+z}{=}((y+z)^2-yz)^2=(y^2+z^2+yz)^2$$

$$=(y^4+2y^3z+3y^2z^2.+2yz^3+z^4)_{\text{using formula } (p+q+r)^2=\sum p^2+2\sum pq} \ (4)$$

$$\sqrt[4]{\frac{(-a^6+b^6+c^6)(-a^{14}+b^{14}+c^{14})}{21(a^4-b^2c^2)^2}}\overset{(2),(3),(4)}{=}$$

$$=\sqrt[4]{\left(\frac{(-3yz(y+z))\left(-7yz(y+z)(y^4+2y^3z+3y^2z^2.+2yz^3+z^4)\right)}{(y^4+2y^3z+3y^2z^2.+2yz^3+z^4)21}\right)}$$

$$=\sqrt[4]{(y+z)^2y^2z^2}\overset{a^2=b^2+c^2 \text{ or } x=y+z}{=}\sqrt[4]{(x)^2y^2z^2}\overset{a^2=x,b^2=y,c^2=z}{=}$$

$$=\sqrt[4]{a^4b^4c^4}=abc=4Rrs\overset{(1)}{=}4Rr(2R+r)$$