ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum \frac{1}{\sqrt{4m_b^2 - h_a^2} + \sqrt{c^2 - h_a^2}} = \sum \frac{a}{a^2 + 2r_ar_c - 2rr_b}$$

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$$\frac{1}{\sqrt{4m_b^2 - h_a^2} + \sqrt{c^2 - h_a^2}} = \frac{\sqrt{4m_b^2 - h_a^2} - \sqrt{c^2 - h_a^2}}{\left(\sqrt{4m_b^2 - h_a^2} + \sqrt{c^2 - h_a^2}\right) \cdot \left(\sqrt{4m_b^2 - h_a^2} - \sqrt{c^2 - h_a^2}\right)} =$$

$$= \frac{\sqrt{4m_b^2 - h_a^2} - \sqrt{c^2 - h_a^2}}{4m_b^2 - c^2} = \frac{\sqrt{4m_b^2 - h_a^2} - \sqrt{c^2 - h_a^2}}{a^2 + (a^2 + c^2 - b^2)}$$

$$\boxed{\begin{array}{l} Lemma\ \ 1:\ ac = r_ar_c + rr_b\ \ (true) \\ Lemma\ \ 2:\ b^2 = (a-c)^2 + 4rr_b\ \ (true) \end{array}}$$

$$From\ Lemma\ 1\ and\ Lemma\ 2,\quad we\ have:$$

$$b^2 = a^2 - 2ac + c^2 + 4rr_b = a^2 + c^2 - 2(r_ar_c + rr_b) + 4rr_b = a^2 + c^2 - 2r_ar_c + 2rr_b$$

$$a^2 + c^2 - b^2 = 2r_ar_c - 2rr_b$$

$$Now\ let\ prove\ that:\ \sqrt{4m_b^2 - h_a^2} - \sqrt{c^2 - h_a^2} = a\ \ (2)$$

$$For\ this:\ \sqrt{4m_b^2 - h_a^2} = a + \sqrt{c^2 - h_a^2}$$

$$4m_b^2 - h_a^2 = a^2 + c^2 - h_a^2 + 2a\sqrt{c - h_a^2}$$

$$4m_b^2 - a^2 - c^2 = 2a\sqrt{c - h_a^2}$$

$$a^2 + c^2 - b^2 = 2a\sqrt{c - h_a^2}$$

$$(a^2 + c^2 - b^2)^2 = 4a^2c^2 - 4a^2h_a^2\ (Because\ \ \Delta ABC\ acute)$$

$$4a^2h_a^2 = 4a^2c^2 - (a^2 + c^2 - b^2)^2 = (2ac - a^2 - c^2 + b^2)(2ac + a^2 + c^2 - b^2) =$$

$$[b^2 - (a-c)^2][(a+c)^2 - b^2] = (b - a + c)(b + a - c)(a + c - b)(a + c + b) =$$

$$(2p-2a)(2p-2c)(2p-2b)2p$$

$$4a^2h_a^2 = 16p(p-a)(p-b)(p-c)$$

$$2ah_a = 4\sqrt{p(p-a)(p-b)(p-c)}$$

$$S = \sqrt{p(p-a)(p-b)(p-c)}\ (true)$$

$$So\ \ (2)\ true$$

$$Analogously,\quad it\ is\ true\ in\ the\ other\ three.$$