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In any scalene Δ ABC, the following relationship holds :

$$\sum_{cyc}\sqrt[4]{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}}=\frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{r_b}{r}}+\sqrt{\frac{r}{r_c}}\right)=\sum_{cyc}\frac{\sqrt{bc}}{w_a}$$

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$$\sqrt[4]{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}}=\sqrt[4]{\frac{s(s-a)+\frac{(b-c)^2}{4}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)}{s(s-a)-\frac{s(s-a)(b-c)^2}{(b+c)^2}-\left(s(s-a)-\frac{s(s-a)(b-c)^2}{a^2}\right)}}$$

$$=\sqrt[4]{\frac{\frac{1}{4}+\frac{s(s-a)}{a^2}}{\frac{s(s-a)}{a^2}-\frac{s(s-a)}{(b+c)^2}}}=\sqrt[4]{\frac{\frac{4s^2-4sa+a^2}{4a^2}}{\frac{s(s-a)}{a^2(b+c)^2}.\left((b+c)^2-a^2\right)}}=\sqrt[4]{\frac{\frac{(2s-a)^2}{4a^2}}{\frac{s(s-a)}{a^2(b+c)^2}.\left(4s(s-a)\right)}}$$

$$=\sqrt[4]{\frac{(b+c)^4}{16s^2(s-a)^2}}\Rightarrow\sqrt[4]{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}}=\frac{1}{2}.\frac{b+c}{\sqrt{s(s-a)}}\ \textit{and analogs}$$

$$\therefore\sum_{cyc}\sqrt[4]{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}}=\frac{1}{2}.\sum_{cyc}\frac{b+c}{\sqrt{s(s-a)}}\to(1)$$

$$\text{Again},\sum_{cyc}\left(\sqrt{\frac{r_b}{r}}+\sqrt{\frac{r}{r_c}}\right)=\sum_{cyc}\left(\sqrt{\frac{r_a}{r}}+\sqrt{\frac{r}{r_a}}\right)=\sum_{cyc}\frac{r_a+r}{\sqrt{rr_a}}=\sum_{cyc}\frac{rs\left(\frac{1}{s-a}+\frac{1}{s}\right)}{\sqrt{(s-b)(s-c)}}$$

$$=\sum_{cyc}\frac{\sqrt{s(s-a)(s-b)(s-c)}.\frac{2s-a}{s(s-a)}}{\sqrt{(s-b)(s-c)}}$$

$$=\sum_{cyc}\frac{b+c}{\sqrt{s(s-a)}}\therefore\frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{r_b}{r}}+\sqrt{\frac{r}{r_c}}\right)=\frac{1}{2}.\sum_{cyc}\frac{b+c}{\sqrt{s(s-a)}}\to(2)\ \textit{and}$$

$$\textit{finally},\sum_{cyc}\frac{\sqrt{bc}}{w_a}=\sum_{cyc}\frac{\sqrt{bc}}{\frac{2\sqrt{bc}}{b+c}.\sqrt{s(s-a)}}\therefore\sum_{cyc}\frac{\sqrt{bc}}{w_a}=\frac{1}{2}.\sum_{cyc}\frac{b+c}{\sqrt{s(s-a)}}\to(3)$$

$$\therefore(1),(2),(3)\Rightarrow\sum_{cyc}\sqrt[4]{\frac{m_a^2-h_a^2}{w_a^2-h_a^2}}=\frac{1}{2}\sum_{cyc}\left(\sqrt{\frac{r_b}{r}}+\sqrt{\frac{r}{r_c}}\right)=\sum_{cyc}\frac{\sqrt{bc}}{w_a}\ (\text{QED})$$