ROMANIAN MATHEMATICAL MAGAZINE

In Δ ABC, prove that the perimeter can be represented by 3 following sums:

$$\sum\sqrt{m_a^2+rr_a^2}=\frac{1}{2}\sum(r_c+r)\sqrt{\frac{r_a+r_b}{r_c-r}}=\sum(a+b)\cos C$$

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Solution by Tapas Das-India

$$\sum(a+b)\cos C=(a+b)\cos C+(b+c)\cos A+(c+a)\cos B=$$

$$=\sum(b\cos C+c\cos B)=\sum a=2s\ (A)$$

$$r_c+r=\frac{rs}{s-c}+r=\frac{r(s+s-c)}{s-c}=\frac{r(2s-c)}{s-c}$$

$$r_a+r_b=\frac{F}{s-a}+\frac{F}{s-b}=\frac{F(2s-a-b)}{(s-a)(s-b)}=\frac{cF}{(s-a)(s-b)}$$

$$r_c-r=\frac{F}{s-c}-r=\frac{r(s-s+c)}{s-c}=\frac{cr}{s-c},\ \frac{r_a+r_b}{r_c-r}=\frac{cF}{(s-a)(s-b)}.\frac{s-c}{cr}=\frac{(s-c)s}{(s-a)(s-b)}$$

$$=\frac{(s-c)^2s}{(s-a)(s-c)(s-b)}=\frac{(s-c)^2s}{sr^2}=\frac{(s-c)^2}{r^2},\ \sqrt{\frac{r_a+r_b}{r_c-r}}=\frac{s-c}{r}$$

$$(r_c+r)\sqrt{\frac{r_a+r_b}{r_c-r}}=\frac{r(2s-c)}{s-c}.\frac{s-c}{r}=2s-c\ and$$

$$\frac{1}{2}\sum(r_c+r)\sqrt{\frac{r_a+r_b}{r_c-r}}=\frac{1}{2}\sum 2s-c=\frac{1}{2}.(4s)=2s\ \ (B)$$

(*Reference: Bogdan Fustei, Mohamed Amine Ben Ajiba about NAGEL cevians*)

$$m_a^2+rr_a=s(s-a)+\frac{(b-c)^2}{4}+\frac{sr^2}{s-a}=s(s-a)+\frac{(b-c)^2}{4}+\frac{sr^2(s-b)(s-c)}{(s-a)(s-b)(s-c)}=$$

$$=s(s-a)+\frac{(b-c)^2}{4}+\frac{sr^2(s-b)(s-c)}{sr^2}=s(s-a)+\frac{(b-c)^2}{4}+(s-b)(s-c)$$

$$=s(s-a)+(s-b)(s-c)+\frac{(b-c)^2}{4}=2s^2-s(a+b+c)+bc+\frac{(b-c)^2}{4}=$$

$$=2s^2-2s^2+\frac{(b-c)^2+4bc}{4}=\frac{(b+c)^2}{4}\ and$$

$$\sum\sqrt{m_a^2+rr_a^2}=\sum\frac{b+c}{2}=a+b+c=2s\quad (C)$$

$$from\ (A),(B),(C)\ we\ get\ \sum\sqrt{m_a^2+rr_a^2}=\frac{1}{2}\sum(r_c+r)\sqrt{\frac{r_a+r_b}{r_c-r}}=\sum(a+b)\cos C$$