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In ΔABC the following relationships holds:

$$\sum \frac{asinA}{2(r_b+r_c)} = \sum \frac{r_a^2}{r_a^2+s^2} = \sum \frac{a(cosB+cosC)}{2(b+c)} = \frac{1}{2}\sum sinA\sqrt{\frac{rr_a}{r_br_c}} = \sum sin^2\frac{A}{2}$$

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$$\frac{asinA}{2(r_b+r_c)} = \frac{asinA}{2S\left(tan\frac{B}{2}+tan\frac{C}{2}\right)} = \frac{a}{2S}\cdot\frac{sinA}{\frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{cos\frac{B}{2}\,.\,cos\frac{C}{2}}} = \frac{a}{2S}\cdot\frac{sinA\,.cos\frac{B}{2}\,.\,cos\frac{C}{2}}{cos\frac{A}{2}} =$$

$$\frac{a}{2S}\cdot\frac{2sin\frac{A}{2}\,.\,cos\frac{A}{2}.cos\frac{B}{2}\,.\,cos\frac{C}{2}}{cos\frac{A}{2}} = \frac{a}{2S}\,.2tan\frac{A}{2}.\frac{S}{4R} = \frac{2RsinA}{4R}.tan\frac{A}{2}$$

$$= \frac{2sin\frac{A}{2}\,.\,cos\frac{A}{2}}{2}.tan\frac{A}{2} = sin^2\frac{A}{2}$$

$$Therefore\ :\ \sum_{cyc}\frac{asinA}{2(r_b+r_c)} = \sum_{cyc}sin^2\frac{A}{2}\quad (1)$$

$$\frac{r_a^2}{r_a^2+s^2} = \frac{s^2tan^2\frac{A}{2}}{s^2tan^2\frac{A}{2}+s^2} = \frac{tan^2\frac{A}{2}}{tan^2\frac{A}{2}+1} = \frac{tan^2\frac{A}{2}}{\frac{1}{cos^2\frac{A}{2}}} = sin^2\frac{A}{2}$$

$$So\ :\ \sum_{cyc}\frac{r_a^2}{r_a^2+s^2} = \sum_{cyc}sin^2\frac{A}{2}\quad (2)$$

$$\frac{a(cosB+cosC)}{2(b+c)} = \frac{2RsinA(cosB+cosC)}{2\,.2R(sinB+sinC)} =$$

$$= \frac{sinA\,.\,2cos\frac{B+C}{2}\,.\,cos\frac{B-C}{2}}{2\,.\,2sin\frac{B+C}{2}\,.\,cos\frac{B-C}{2}} = \frac{1}{2}sinA.ctg\frac{B+C}{2} =$$

$$= \frac{1}{2}sinA\,.\,tan\frac{A}{2} = \frac{1}{2}\,.2sin\frac{A}{2}.cos\frac{A}{2}.tan\frac{A}{2} = sin^2\frac{A}{2}$$

$$So\ :\ \sum_{cyc}\frac{a(cosB+cosC)}{2(b+c)}=\sum_{cyc}sin^2\frac{A}{2}\quad (3)$$

$$\frac{1}{2}sinA\sqrt{\frac{rr_a}{r_br_c}}=\frac{1}{2}sinA\left(\frac{r\,.\,S\,.\,tan\frac{A}{2}}{S\,.\,tan\frac{B}{2}.S\,.\,tan\frac{C}{2}}\right)^{\frac{1}{2}}=\frac{1}{2}sinA(\frac{r}{S}\,.\,\frac{tan^2\frac{A}{2}}{tan\frac{A}{2}.tan\frac{B}{2}.tan\frac{C}{2}})^{\frac{1}{2}}=$$

$$\frac{1}{2}sinA\left(\frac{r}{S}.\frac{tan^2\frac{A}{2}}{\frac{r}{S}}\right)^{\frac{1}{2}}=\frac{1}{2}sinA\,.\,tan\frac{A}{2}=sin^2\frac{A}{2}$$

$$Also\ :\ \frac{1}{2}\sum_{cyc}sinA\sqrt{\frac{rr_a}{r_br_c}}=\sum_{cyc}sin^2\frac{A}{2}\quad (4)\qquad (Proved)$$