ROMANIAN MATHEMATICAL MAGAZINE

For non-right triangle ABC the following relationship holds:

$$\prod_{cyc}\sqrt[6]{\frac{\sin^2 2A}{c^2-h_a^2}}=\sum_{cyc}\frac{\cos A+\cos B}{2r_c}=\left(\sum_{cyc}\sqrt{\frac{r_a-r}{(r_a+r_b)(r_a+r_c)}}\right)^2=\frac{1}{R}$$

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Solution by Tapas Das-India

$$\sin^2 2A=4\sin^2 A\,(1-\sin^2 A)=\frac{4a^2}{4R^2}\left(1-\frac{a^2}{4R^2}\right)=\frac{a^2}{4R^4}(4R^2-a^2)\ and$$

$$c^2-h_a^2=c^2-\frac{b^2c^2}{4R^2}=\frac{c^2}{4R^2}(4R^2-b^2)$$

$$\prod_{cyc}\sqrt[6]{\frac{\sin^2 2A}{c^2-h_a^2}}=\prod_{cyc}\sqrt[6]{\frac{\frac{a^2}{4R^4}(4R^2-a^2)}{\frac{c^2}{4R^2}(4R^2-b^2)}}=\prod_{cyc}\sqrt[6]{\frac{a^2(4R^2-a^2)}{R^2c^2(4R^2-b^2)}}=\prod\sqrt[6]{\frac{a^2b^2c^2}{R^6a^2b^2c^2}}=\frac{1}{R}$$

$$\sum_{cyc}\frac{\cos A+\cos B}{2r_c}=\sum\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2s\tan\frac{C}{2}}=\frac{1}{2s}\sum\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2s\tan\frac{C}{2}}=$$

$$=\frac{1}{2s}\sum\frac{2\sin\frac{C}{2}\cos\frac{A-B}{2}}{2s\tan\frac{C}{2}}=\sum\frac{2\cos\frac{C}{2}\cos\frac{A-B}{2}}{2s}=$$

$$=\frac{1}{2s}\sum 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}=\frac{1}{2s}\sum(\sin A+\sin B)=\frac{2}{2s}.\frac{s}{R}=\frac{1}{R}$$

$$r_a-r=r\left(\frac{s}{s-a}-1\right)=\frac{ar}{s-a}$$

$$r_a+r_b=F\left(\frac{1}{s-a}+\frac{1}{s-b}\right)=\frac{F(2s-a-b)}{(s-a)(s-b)}=\frac{Fc}{(s-a)(s-b)}\ and$$

$$r_a+r_c=\frac{bF}{(s-a)(s-c)},\ using\ above\ result\ we\ get$$

$$\frac{r_a-r}{(r_a+r_b)(r_a+r_c)}=\frac{ar}{bcs}=\frac{a^2r}{abcs}=\frac{a^2}{4Rs^2}$$

$$\left(\sum_{cyc}\sqrt{\frac{r_a-r}{(r_a+r_b)(r_a+r_c)}}\right)^2=\left(\sum\frac{a}{2s\sqrt{R}}\right)^2=\frac{\frac{1}{R}(a+b+c)^2}{4s^2}=\frac{1}{R}$$