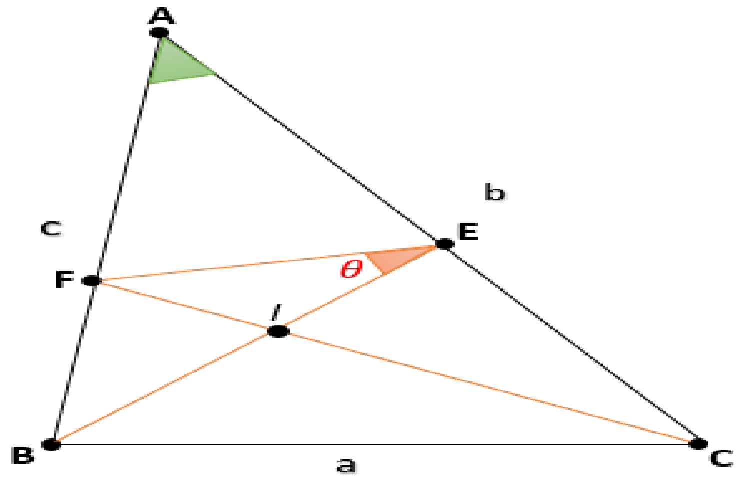
ROMANIAN MATHEMATICAL MAGAZINE



I-incenter.Prove that :

$$\cot\theta=\frac{1}{\sin A}\cdot\left[2-\cos A+\frac{b}{a+c}(1-2\cos A)\right]$$

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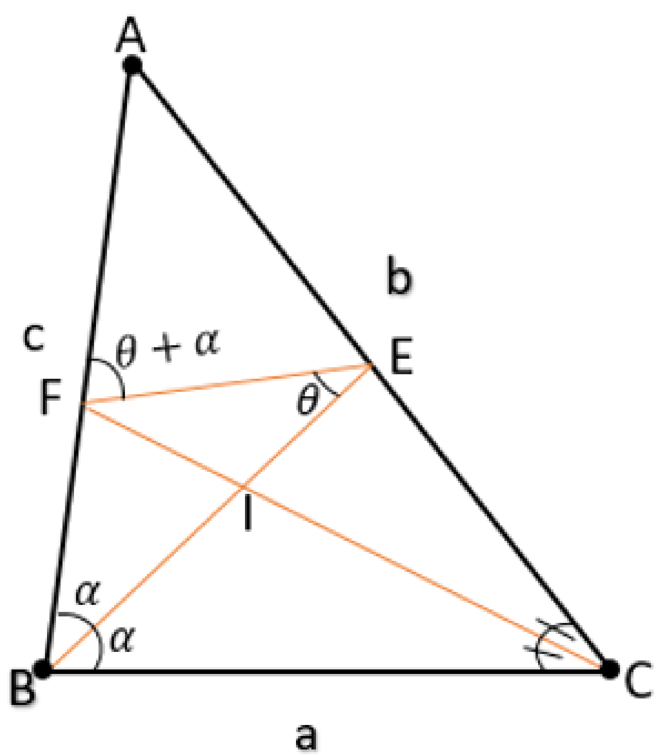

İn ΔABC BE and CF are the bisectors. Then:

$$AE=\frac{bc}{a+c}\ ;\ AF=\frac{bc}{a+b}\ ;BF=\frac{ac}{a+b}\quad (1)$$

İn ΔAFE rule cosine:

$$FE^2=AF^2+AE^2-2AF\cdot AE\cos A\ \ (2)$$

İn ΔABE rule cosine:

$$BE^2=AB^2+AE^2-2AB\cdot AE\cos A\ =$$

$$=(AF+FB)^2+AE^2-2(AF+FB)\cdot AE\cos A\ \ (2A)$$

İn ΔFEB rule sine :

$$\frac{\sin\theta}{BF}=\frac{\sin B\widehat{F}E}{BE}\Rightarrow BE=BF\cdot\frac{\sin(\theta+\alpha)}{\sin\theta}\ (3)$$

İn ΔAFE rule sine :

$$\frac{\sin(\theta+\alpha)}{AE}=\frac{\sin A}{FE}\Rightarrow FE=AE\cdot\frac{\sin A}{\sin(\theta+\alpha)}\quad (4)$$

From (3) and (4) we have :

$$BE\cdot FE=BF\cdot AE\cdot\frac{\sin A}{\sin\theta}\quad (5)$$

İn ΔFEB rule cosine :

$$\cos\theta=\frac{FE^2+BE^2-BF^2}{2FE\cdot BE}\quad (*)$$

Let's use the expressions (2),(2A) and (5) in $(*)$

$$\cos\theta=\frac{AF^2+AE^2-2AF\cdot AE\cos A\ +(AF+FB)^2+AE^2-2(AF+FB)\cdot AE\cos A}{2BF\cdot AE\cdot\frac{\sin A}{\sin\theta}}$$

$$\cot(\theta)\sin(A)=\frac{AF^2}{BF\,AE}+\frac{AE}{BF}+\frac{AF}{AE}-\frac{2AF\cos(A)}{BF}-\cos(A)$$

$$Let's\ use\ expression\ \ (1)$$

$$\cot(\theta)\sin(A)=\left(\frac{bc}{a+b}\right)^2\frac{a+b}{ac}\frac{a+c}{b+c}+\frac{bc}{a+c}\frac{a+b}{ac}+\frac{bc}{a+b}\frac{a+c}{bc}-\frac{2bc}{a+b}\frac{a+b}{ac}\cos(A)-\cos(A))$$

$$=\frac{(a+c)b}{(a+b)a}+\frac{b(a+b)}{a(a+c)}+\frac{a+c}{a+b}-\frac{2b}{a}\cos(A)-\cos(A))$$

$$=\frac{a+c}{a+b}\left(\frac{b}{a}+1\right)+\frac{b(a+b)}{a(a+c)}-\frac{2b}{a}\cos(A)-\cos(A)=\frac{a+c}{a}+\frac{b(a+b)}{a(a+c)}-\frac{2b}{a}\cos(A)-\cos(A)$$

$$=\left((1-\cos(A))+\frac{c}{a}+\frac{b(a+b)}{a(a+c)}-\frac{2b}{a}\cos(A)\right)$$

$$=1-\cos(A)+\frac{c(a+c)+b(a+b)}{a(a+c)}-\frac{2b}{a}\cos(A)=$$

$$=1-\cos A+\frac{ac+c^2+ab+b^2}{a(a+c)}-\frac{2b}{a}\cdot\cos A=$$

$$=1-\cos(A)+\frac{ac+ab+a^2+2bc\cos(A)}{a(a+c)}-\frac{2b}{a}\cos(A)$$

$$=1-\cos(A)+1+\frac{b(a+2c\cos(A))}{a(a+c)}-\frac{2b}{a}\cos(A)$$

$$=2-\cos(A)+\frac{b}{a+c}\left(\frac{a+2\cos(A)}{a}-\frac{2(a+c)\cos(A)}{a}\right)$$

$$=2-\cos(A)+\frac{b}{a+c}\frac{a-2a\cos(A)}{a}=2-\cos(A)+\frac{b}{a+c}(1-2\cos(\mathrm{A}))$$

$$\cot(\theta)=\frac{1}{\sin(A)}\left(2-\cos(A)+\frac{b}{a+c}(1-2\cos(A))\right)\ \ (proved)$$