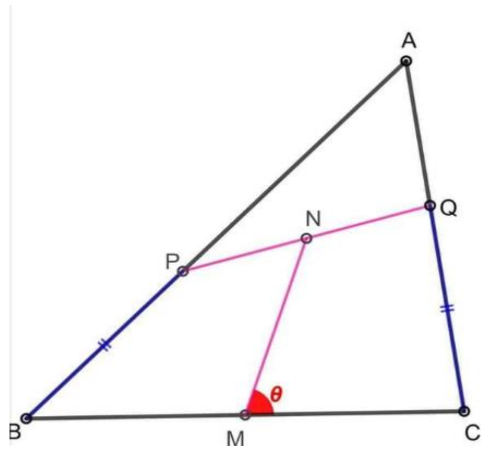
ROMANIAN MATHEMATICAL MAGAZINE

$M, N -$ *are midpoints. Prove :*

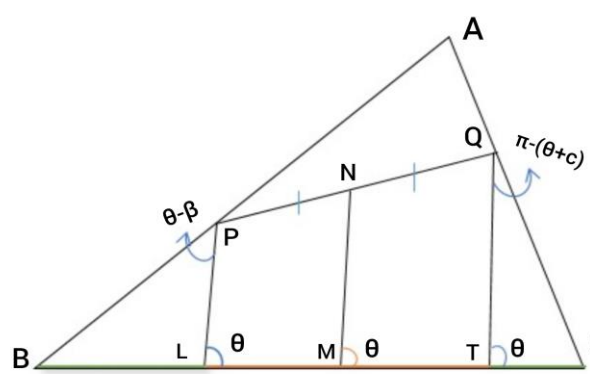
$$\cot\theta = \frac{b\cdot\cot B + c\cdot\cot A - b\cdot\csc A}{b+c}$$

$$\text{If } A = \frac{\pi}{2} \text{ then } \cot\theta = \frac{c-b}{c+b} = \frac{\cot B - 1}{\cot B + 1} = \cot\left(B + \frac{\pi}{4}\right)$$



Proposed by Thanasis Gakopoulos-Greece, Istvan Biro-Romania

Solution by Mirsadix Muzefferov-Azerbaijan



$$\text{By construction } PL \parallel NM \parallel QT. \text{ According to the given conditions.}$$

$$BP = QC\ ;\ PN = NQ\ ;\ BM = MC$$

$$\text{Then, according to Thales' theorem } LM = MT.$$

$$\text{Therefore } BL = TC$$

$$\left.\begin{array}{l} \text{In } \Delta BPL \text{ rule sine } \dfrac{\sin(\theta - B)}{BL} = \dfrac{\sin\theta}{BP} \\ \text{In } \Delta CQT \text{ rule sine } \dfrac{\sin(\theta + C)}{TC} = \dfrac{\sin\theta}{QC} \end{array}\right\} \Rightarrow \sin(\theta - B) = \sin(\theta + C)\ (*)$$

$$(*) \Rightarrow sin(\theta - B)\cdot sin\,A = sin(\theta + C)\ \cdot sin\,A \Rightarrow \frac{1}{2}\left(cos\left(\theta - (A+B)\right) - cos(\theta + A - B)\right) =$$

$$= \frac{1}{2}\left(cos(\theta + C - A) - cos\left(\theta + (A+C)\right)\right)$$

$$cos\left(\theta - (A+B)\right) - cos(\theta + A - B) = cos(\theta + C - A) - cos\left(\theta + (\pi - B)\right)$$

$$cos(A + B - \theta) - cos\left(\theta + (A-B)\right) = cos\left(\theta + (C-A)\right) - cos\left(\pi + (\theta - B)\right)$$

$$cos(A + B - \theta) - cos\left(\theta + (A-B)\right) = cos\left(\theta + (C-A)\right) - cos(\theta - B)$$

$$Let's\ add\ (-\,cos(B+\theta))\ to\ both\ sides$$

$$cos(A+B-\theta) - cos\left(\theta + (A-B)\right) - cos(B+\theta) - cos\left(\theta + (C-A)\right) =$$

$$= cos(\theta - B) - cos(B + \theta);$$

$$Here: -cos(B+\theta) = -\cos(\pi - (A+C) + \theta)\, cos(A + C - \theta)$$

$$\left(cos(A+B-\theta) - cos\left(\theta + (A-B)\right)\right) + \left(cos(A+C-\theta) - cos\left(\theta + (C-A)\right)\right) =$$

$$= cos(B - \theta) - cos(B + \theta)$$

$$-2\,sin\frac{(A+B-\theta) - \left(\theta + (A-B)\right)}{2}\cdot sin\frac{(A+B-\theta) + (\theta + A - B)}{2} -$$

$$-2\,sin\frac{(A+C-\theta) - \left(\theta + (C-A)\right)}{2}\cdot sin\frac{(A+C-\theta) + (\theta + C - A)}{2} =$$

$$= -2sin\frac{(B-\theta) - (B+\theta)}{2}\cdot \sin\frac{(B-\theta) + (B+\theta)}{2}$$

$$sin(B-\theta)\cdot sin\,A + sin(A-\theta)\cdot sin\,C = -\,sin\,\theta\cdot sin\,B$$

$$Multiply\ both\ sides\ by\ (2R\,sin\,B)$$

$$2R\,sin\,B\cdot sin(\theta - B)\cdot sin\,A + 2R\,sin\,B\cdot sin(\theta - A)\,sin\,C = 2R\,sin\,B\,sin\,\theta\,sin\,B \Rightarrow$$

$$\Rightarrow b\,sin(\theta - B)\cdot sin\,A + c\,sin\,B\cdot sin(\theta - A) = b\,sin\,B\,sin\,\theta$$

$$b\,sin\,A(sin\,\theta\,cos\,B - sin\,B\,cos\,\theta) + c\,sin\,B(sin\,\theta\,cos\,A - sin\,A\,cos\,\theta) = b\,sin\,B\,sin\,\theta$$

$$-b\,sin\,A\,sin\,B\,cos\,\theta - c\,sin\,A\,sin\,B\,cos\,\theta + b\,sin\,A\,cos\,B\,sin\,\theta$$
$$+\,c\,sin\,B\,cos\,A\,sins\,\theta = b\,sin\,B\,sin\,\theta$$

$$(b+c)\,sin\,A\,sin\,B\,cos\,\theta = b\,cos\,B\,sin\,A\,sin\,\theta + c\,cos\,A\,sin\,B\,sin\,\theta - b\,sin\,B\,sin\,\theta$$

$$Let's\ divide\ both\ sides\ by: (sin\,\theta\,sin\,A\,sin\,B)$$

$$(b + C)\,cot\,\theta = b\,cot\,B + c\,cot\,A - \frac{b}{sin\,A}$$

$$cot\,\theta = \frac{b\,cot\,B + c\,cot\,A - b\,csc\,A}{b+c}\quad (proved)$$

$$in\ special\ case\ if\ A = \frac{\pi}{2}\ \Rightarrow\ cot\,B = \frac{c}{b} \Rightarrow cot\,\theta = \frac{b\,cot\,B + c\,cot\,A - b\,csc\,A}{b+c} =$$

$$= \frac{b\cdot\frac{c}{b} + c\cdot\cot\frac{\pi}{2} - b\csc\frac{\pi}{2}}{b+c} = \frac{c-b}{c+b} = \frac{\frac{c}{b} - 1}{\frac{c}{b} + 1} = \frac{cot\,B - 1}{cot\,B + 1} = cot\left(B + \frac{\pi}{4}\right)$$