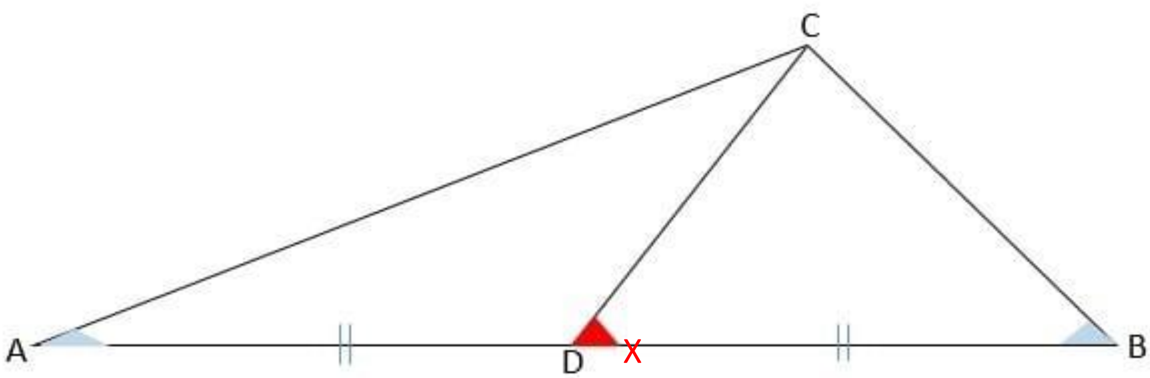
ROMANIAN MATHEMATICAL MAGAZINE



Prove that:

$$\frac{\cos X}{\sin(X+B)}=\frac{\tan B-\tan A}{\tan B+\tan A}\cdot\frac{1}{\sin B}$$

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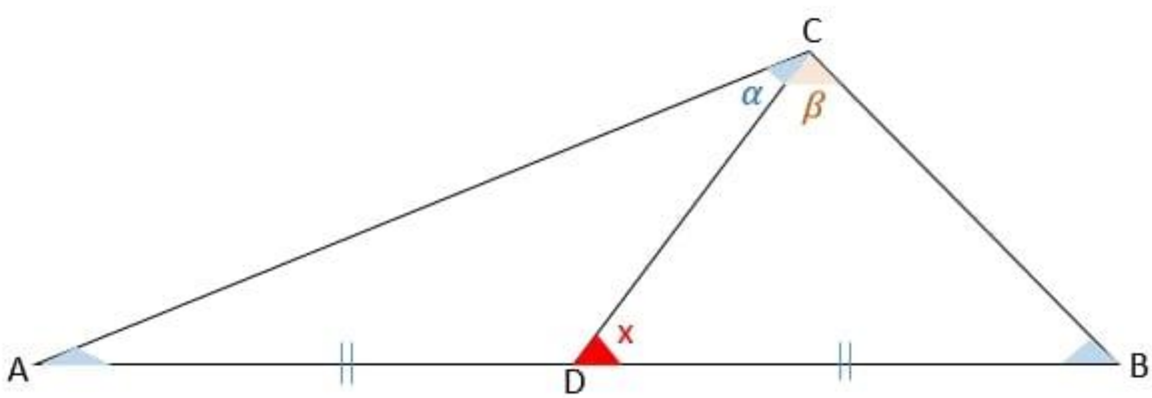

$$\left.\begin{aligned}&In\ \Delta ACD\ rule\ sine:\ \frac{\sin\alpha}{AD}=\frac{\sin A}{DC}\\&In\ \Delta BCD\ rule\ sine:\frac{\sin\beta}{DB}=\frac{\sin B}{DC}\end{aligned}\right\}\Rightarrow$$

$$\Rightarrow\frac{\sin\alpha}{AD}:\frac{\sin\beta}{DB}=\frac{\sin A}{DC}:\frac{\sin B}{DC}\Rightarrow\frac{\sin\alpha}{\sin\beta}=\frac{\sin A}{\sin B}$$

$$\alpha = x - A\,;\, \beta = 180^\circ - (x + B)$$

$$\frac{\sin(x-A)}{\sin(x+B)} = \frac{\sin A}{\sin B} \Rightarrow \frac{\sin x\cos A - \sin A\cos x}{\sin B\cos x + \sin x\cos B} = \frac{\sin A}{\sin B}$$

$$\sin x\cdot\cos A\cdot\sin B - \cos x\cdot\sin A\cdot\sin B = \sin A\cdot\sin x\cdot\cos B + \sin B\cdot\cos x\cdot\sin A$$

$$2\sin A\cdot\sin B\cdot\cos x = \sin(B-A)\cdot\sin x$$

$$\cos x = \frac{\sin B\cos A - \sin A\cos B}{2\sin A\cdot\sin B}\cdot\sin x = \frac{\tan B - \tan A}{2\tan A\tan B}\cdot\sin x \quad (*)$$

$$\begin{matrix} In\ \Delta ABC\ rule\ sine\ \dfrac{AB}{\sin(A+B)} = \dfrac{AC}{\sin B} \\ In\ \Delta DBC\ rule\ sine\ \dfrac{DB}{\sin\beta} = \dfrac{DC}{\sin B} \end{matrix} \Rightarrow \begin{cases} \dfrac{2DB}{\sin(A+B)} = \dfrac{AC}{\sin B} \\ \dfrac{\sin\beta}{DB} = \dfrac{\sin B}{DC} \end{cases} \Rightarrow$$

$$\Rightarrow \frac{2\sin\beta}{\sin(A+B)} = \frac{AC}{DC} \quad (1)$$

$$In\ \Delta ACD \Rightarrow \frac{AC}{DC} = \frac{\sin x}{\sin A} \quad (2)$$

From (1) *and* (2) *we have*

$$\frac{2\sin\beta}{\sin(A+B)} = \frac{\sin x}{\sin A} \Rightarrow \frac{\sin(A+B)}{2\sin(x+B)} = \frac{\sin A}{\sin x} \quad (**)$$

From $(*)$ *and* $(**)$ *we have*

$$\begin{cases} \dfrac{\sin(A+B)}{2\sin(x+B)} = \dfrac{\sin A}{\sin x} \\ \cos x = \dfrac{\tan B - \tan A}{2\tan A\tan B}\cdot\sin x \end{cases} \Rightarrow$$

$$\frac{\sin(A+B)}{2\sin(x+B)}\cdot\cos x = \frac{\sin A}{\sin x}\cdot\frac{\tan B - \tan A}{2\tan A\tan B}\cdot\sin x \Rightarrow$$

$$\Rightarrow \frac{\cos x}{\sin(x+B)} = \frac{\tan B - \tan A}{\dfrac{\sin A\cos B + \sin B\cos A}{\cos A\cdot\cos B}}\cdot\frac{1}{\sin B} \Rightarrow \frac{\cos x}{\sin(x+B)} =$$

$$= \frac{\tan B - \tan A}{\tan B + \tan A}\cdot\frac{1}{\sin B} \quad (proved)$$