ROMANIAN MATHEMATICAL MAGAZINE

In any cyclic quadrilateral, the following relationship holds:

$$\frac{r_a^4}{a^7}+\frac{r_b^4}{b^7}+\frac{r_c^4}{c^7}+\frac{r_d^4}{d^7}\ge\frac{2}{s^3}$$

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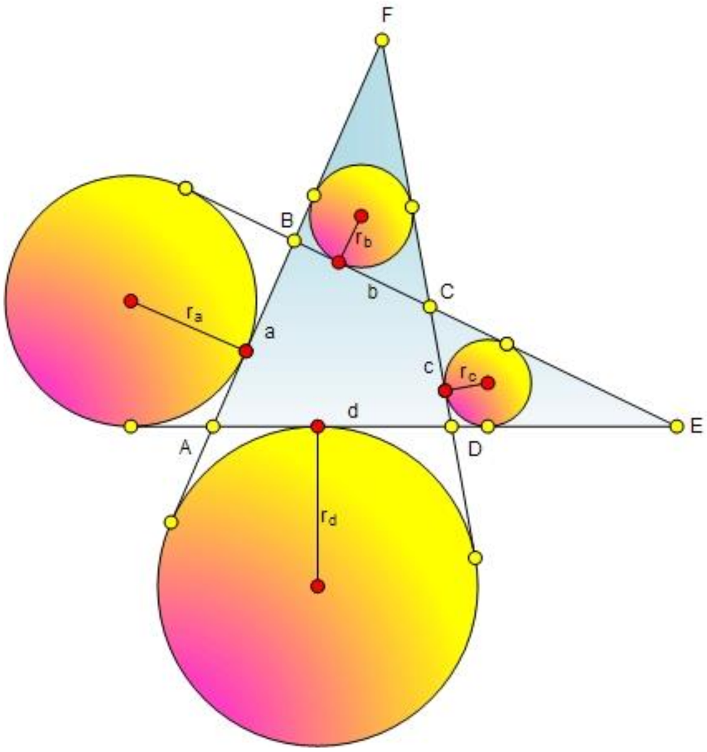

Via Ptolemy's theorem 1 and Ptolemy's theorem 2, we have: $pq = ac + bd$ and

$$\frac{p}{q}=\frac{ad+bc}{ab+cd}\text{ and consequently, } AC = p = \sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}$$

$$\text{Now, }\cos B=\frac{a^2+b^2-\frac{(ac+bd)(ad+bc)}{ab+cd}}{2ab}\Rightarrow 2\cos^2\frac{B}{2}=\frac{a^2+b^2-\frac{(ac+bd)(ad+bc)}{ab+cd}}{2ab}+1$$

$$=\frac{(a^2+b^2)(ab+cd)-(ac+bd)(ad+bc)+2ab(ab+cd)}{2ab(ab+cd)}$$

$$=\frac{(a+b+c-d)(a+b+d-c)}{2ab(ab+cd)}\Rightarrow 1+\tan^2\frac{B}{2}=\frac{4ab(ab+cd)}{(a+b+c-d)(a+b+d-c)}$$

$$\Rightarrow\tan^2\frac{B}{2}=\frac{4ab(ab+cd)-(a+b+c-d)(a+b+d-c)}{(a+b+c-d)(a+b+d-c)}$$

$$=\frac{(c+d+a-b)(c+d+b-a)}{(a+b+c-d)(a+b+d-c)}\Rightarrow\tan^2\frac{B}{2}=\frac{(s-b)(s-a)}{(s-c)(s-d)}$$

$$\text{and analogously}, \tan^2\frac{A}{2}=\frac{(s-a)(s-d)}{(s-b)(s-c)}, \tan^2\frac{C}{2}=\frac{(s-c)(s-b)}{(s-d)(s-a)}$$

$$\text{and } \tan^2\frac{D}{2}=\frac{(s-d)(s-c)}{(s-a)(s-b)}\to(1)$$

Let a perpendicular (r_a) be dropped from the center of the circle with radius r_a onto AB intersecting AB at X.

$$\text{Now}, \tan\left(90^\circ-\frac{A}{2}\right)=\frac{r_a}{AX}\Rightarrow AX=r_a\tan\frac{A}{2} \text{ and } \tan\left(90^\circ-\frac{B}{2}\right)=\frac{r_a}{BX}$$

$$\Rightarrow BX=r_a\tan\frac{B}{2}\Rightarrow AX+BX=a=r_a\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)\overset{\text{via }(1)}{=}$$

$$r_a\left(\sqrt{\frac{(s-a)(s-d)}{(s-b)(s-c)}}+\sqrt{\frac{(s-b)(s-a)}{(s-c)(s-d)}}\right)$$

$$=\frac{r_a(s-a)(s-d+s-b)}{\sqrt{(s-a)(s-b)(s-c)(s-d)}}=\frac{r_a(s-a)(c+a)}{F}$$

$$\Rightarrow r_a=\frac{aF}{(s-a)(c+a)} \text{ and analogously}, r_b=\frac{bF}{(s-b)(b+d)},$$

$$r_c=\frac{cF}{(s-c)(c+a)} \text{ and } r_d=\frac{dF}{(s-d)(b+d)}\to(2)$$

$$\text{Via }(2), \frac{r_a^4}{a^7}+\frac{r_b^4}{b^7}+\frac{r_c^4}{c^7}+\frac{r_d^4}{d^7}$$

$$=F^4\left(\frac{\left(\frac{1}{(s-a)(c+a)}\right)^4}{a^3}+\frac{\left(\frac{1}{(s-b)(b+d)}\right)^4}{b^3}+\frac{\left(\frac{1}{(s-c)(c+a)}\right)^4}{c^3}+\frac{\left(\frac{1}{(s-d)(b+d)}\right)^4}{d^3}\right)$$

$$\overset{\text{Radon}}{\geq}\frac{F^4}{(a+b+c+d)^3}\left(\begin{array}{c}\left(\frac{1}{(s-a)(c+a)}+\frac{1}{(s-c)(c+a)}\right)\\+\left(\frac{1}{(s-b)(b+d)}+\frac{1}{(s-d)(b+d)}\right)\end{array}\right)^4$$

$$=\frac{F^4}{8s^3}\left(\frac{b+d}{(c+a)(s-a)(s-c)}+\frac{c+a}{(b+d)(s-b)(s-d)}\right)^4$$

$$\overset{A-G}{\geq}\frac{F^4}{8s^3}\left(2\sqrt{\frac{b+d}{(c+a)(s-a)(s-c)}\cdot\frac{c+a}{(b+d)(s-b)(s-d)}}\right)^4=\frac{F^4}{8s^3}\cdot\left(\frac{2}{F}\right)^4=\frac{2}{s^3}$$

$$\therefore \frac{r_a^4}{a^7}+\frac{r_b^4}{b^7}+\frac{r_c^4}{c^7}+\frac{r_d^4}{d^7}\geq\frac{2}{s^3}, ''='' \text{ iff ABCD is } a \text{ square (QED)}$$