ROMANIAN MATHEMATICAL MAGAZINE

In any Δ ABC, the following relationship holds :

$$\left(\frac{m_b h_b}{\sin A}\right)^2+\left(\frac{m_c h_c}{\sin B}\right)^2+\left(\frac{m_a h_a}{\sin C}\right)^2 \ge \frac{16F^3}{\sqrt{3}}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\left(\frac{m_b h_b}{\sin A}\right)^2+\left(\frac{m_c h_c}{\sin B}\right)^2+\left(\frac{m_a h_a}{\sin C}\right)^2=\sum_{cyc}\left(\frac{m_a h_a}{\sin C}\right)^2$$

$$=\sum_{cyc}\frac{(2b^2+2c^2-a^2)b^2c^2}{16R^2.\frac{c^2}{4R^2}}=\sum_{cyc}\frac{2b^4+2b^2c^2-a^2b^2}{4}=\frac{1}{4}\left(2\sum_{cyc}a^4+\sum_{cyc}a^2b^2\right)$$

$$\Rightarrow LHS=\frac{1}{4}\left(5\sum_{cyc}a^2b^2-32r^2s^2\right)\to(1)$$

$$\text{Again},\frac{16F^3}{\sqrt{3}}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=\frac{16r^3s^3\left(\sum_{cyc}a^2b^2\right)}{\sqrt{3}.16R^2r^2s^2}=\frac{rs\left(\sum_{cyc}a^2b^2\right)}{\sqrt{3}.R^2}\overset{\text{Mitrinovic}}{\le}$$

$$\frac{r.3\sqrt{3}.R\left(\sum_{cyc}a^2b^2\right)}{2.\sqrt{3}.R^2}=\frac{3r\sum_{cyc}a^2b^2}{2R}\overset{?}{\le}\frac{1}{4}\left(5\sum_{cyc}a^2b^2-32r^2s^2\right)$$

$$\Leftrightarrow R\left(5\sum_{cyc}a^2b^2-32r^2s^2\right)\overset{?}{\ge}6r\sum_{cyc}a^2b^2\Leftrightarrow(5R-6r)\sum_{cyc}a^2b^2\underset{(*)}{\overset{?}{\ge}}32Rr^2s^2$$

$$\text{Now},(5R-6r)\sum_{cyc}a^2b^2\ge(5R-6r)abc\sum_{cyc}a=8Rrs^2(5R-6r)\overset{?}{\ge}32Rr^2s^2$$

$$\Leftrightarrow 5R-6r\overset{?}{\ge}4r\Leftrightarrow 5(R-2r)\overset{?}{\ge}0\to\text{true}\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{16F^3}{\sqrt{3}}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\le\frac{1}{4}\left(5\sum_{cyc}a^2b^2-32r^2s^2\right)\overset{\text{via }(1)}{=}$$

$$\left(\frac{m_b h_b}{\sin A}\right)^2+\left(\frac{m_c h_c}{\sin B}\right)^2+\left(\frac{m_a h_a}{\sin C}\right)^2,''='' \text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

Solution 2 by proposer

$$\sum_{cyc}\left(\frac{m_b h_b}{\sin < A}\right)^2=\sum_{cyc}\left(\frac{4FRm_b}{ab}\right)^2\ge\sum_{cyc}\left(\frac{4FR}{ab}\times\frac{a^2+c^2}{4R}\right)^2=F^2\sum_{cyc}\left(\frac{a^2+c^2}{ab}\right)^2$$

$$F^2\sum_{cyc}\left(\frac{a^2+c^2}{ab}\right)^2\ge^{AM-GM}F^2\sum_{cyc}\left(\frac{2ac}{ab}\right)^2=4F^2\sum_{cyc}\frac{c^2}{b^2}$$

$$4F^2\sum_{cyc}\frac{c^2}{b^2}=4F^2\sum_{cyc}\left(\frac{\frac{c^2}{b^2}+\frac{c^2}{b^2}+\frac{a^2}{c^2}}{3}\right)\ge^{AM-GM}4F^2\sum_{cyc}\sqrt[3]{\frac{a^2c^2}{b^4}}=4F^2\sum_{cyc}\sqrt[3]{\frac{(abc)^2}{b^6}}$$

$$4F^2\sqrt[3]{(abc)^2}\sum_{cyc}\frac{1}{a^2}\ge 4F^2\times\frac{4F}{\sqrt{3}}\sum_{cyc}\frac{1}{a^2}=\frac{16F^3}{\sqrt{3}}\sum_{cyc}\frac{1}{a^2}$$

$$\left(\frac{m_bh_b}{sin<A}\right)^2+\left(\frac{m_ch_c}{sin<B}\right)^2+\left(\frac{m_ah_a}{sin<C}\right)^2\ge\frac{16F^3}{\sqrt{3}}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)$$

Note Section :

$$sin<A=\frac{a}{2R}\qquad h_b=\frac{2F}{b}$$

$$m_a\ge\frac{b^2+c^2}{4R}\qquad\sqrt[3]{(abc)^2}\ge\frac{4F}{\sqrt{3}}$$