ROMANIAN MATHEMATICAL MAGAZINE

**In any $\Delta$ ABC, the following relationship holds :**

$$\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}+\frac{(r+4R)^2\left(6s-\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\right)}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}\le\frac{3R}{2r}$$

*Proposed by Adil Abdullayev-Azerbaijan*

## Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}+\frac{(r+4R)^2\left(6s-\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\right)}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}$$

$$=\frac{(4R+r)^2}{s^2}+\frac{6(4R+r)^2}{s\left(2s+2\sum_{cyc}\sqrt{ab}\right)}-\frac{(4R+r)^2}{s^2}\overset{GM-HM}{\le}\frac{6(4R+r)^2}{s\left(2s+4\sum_{cyc}\frac{ab}{a+b}\right)}$$

$$=\frac{3(4R+r)^2}{s\left(s+2.4Rrs.\sum_{cyc}\frac{1}{ca+bc}\right)}\overset{Bergstrom}{\le}\frac{3(4R+r)^2}{s\left(s+2.4Rrs..\frac{9}{2\sum_{cyc}ab}\right)}=\frac{3(4R+r)^2}{s^2+\frac{36Rrs^2}{s^2+4Rr+r^2}}\overset{?}{\le}\frac{3R}{2r}$$

$$\Leftrightarrow Rs^4+rs^2(8R^2-15Rr-2r^2)\underset{(*)}{\overset{?}{\ge}}2r^2(4R+r)^3$$

$$\text{Now, LHS of }(*)\overset{Gerretsen}{\ge}\left(R(16Rr-5r^2)+r(8R^2-15Rr-2r^2)\right)s^2$$

$$\overset{Gerretsen}{\ge}r(24R^2-20Rr-2r^2)(16Rr-5r^2)\overset{?}{\ge}2r^2(4R+r)^3$$

$$\Leftrightarrow 64t^3-134t^2+11t+2\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(60t^2+3t(t-2)+(t^2-4)+3\right)\overset{?}{\ge}0\to\text{true}\because t\overset{Euler}{\ge}2\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{(r_a+r_b+r_c)^2}{r_ar_b+r_br_c+r_cr_a}+\frac{(r+4R)^2\left(6s-\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\right)}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}\le\frac{3R}{2r}\ \forall\Delta ABC,$$

$''=''$ iff $\Delta$ ABC is equilateral (QED)

## Solution 2 by Mohamed Amine Ben Ajiba-Tanger-Morocco

Since $r_a+r_b+r_c=4R+r$ and $r_ar_b+r_br_c+r_cr_a=s^2$, then we have

$$LHS=\frac{(4R+r)^2}{s^2}+\frac{(r+4R)^2\left(6s-\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\right)}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}=\frac{6s(4R+r)^2}{s^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}$$

$$\overset{Euler}{\le}\frac{4.6s\left(4R+\frac{R}{2}\right)^2}{(a+b+c)^2\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}\overset{AM-GM}{\le}\frac{486sR^2}{\left(3\sqrt[3]{abc}\right)^2\left(3\sqrt[6]{abc}\right)^2}=\frac{6sR^2}{abc}=\frac{3R}{2r},$$

as desired. Equality holds iff $\Delta ABC$ is equilateral.